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The Bounded Approximate Fixed Point Property and Dense Subsets in Banach Spaces and Applications

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Abstract

In this paper, we will first prove that if a Banach space has approximate fixed point property, then its dense subset also has the same property. Also, we introduce the bounded approximate fixed points property for self maps on Banach space. As application, we will prove the existence of fixed points and approximate eigenvalue to certain type of nonexpansive mappings using the existence of bounded approximate fixed points to these maps.

Keywords

ε-fixed Points, Bounded ε-fixed Point, Demi-closed Map, Semi-compact Maps, Demi-compact Maps

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1. Introduction

Fixed point theory is a very popular tool in solving existence problems in many branches of Mathematical Analysis and its applications. Nowadays, there are plenty of problems in applied mathematics which can be solved by means of fixed point theory such as physics, chemistry and economics. In physics and engineering fixed point technique has been used in areas like image retrieval, signal processing and the study of existence and uniqueness of solutions for a class of nonlinear integral equations. But in some situations it will be difficult to derive conditions for existence of fixed points for certain types of mappings and the requirement will be only an approximate to the fixed points. In such a case naturally we will use the concept of approximate fixed points. Some recent work on boundary value problems can be found in [12, 16, 17] and the references therein.

In 1965, F. E. Browder [1] and D. Göhde [8] independently proved that every nonexpansive self-mapping of a closed convex and bounded subset of a uniformly convex Banach space has a fixed point. Also, This result was also obtained by W. A. Kirk [10], under assumptions slightly weaker in a technical sense, and another proof, more geometric and elementary in nature, has recently been given by K. Goebel [6].

In 2001 R. Espinola and W. A. Kirk [4] shown that the product space $H = (M \times K)_\infty$ has the approximate fixed point property for nonexpansive mappings whenever M is a metric space which has the approximate fixed point property for such mappings and K is a bounded convex subset of a Banach space.

In this paper, starting from the article of Mohsenalhosseini [11] and Chakraborty et al [2], we study we approximate fixed point property and its dense subset property on Banach spaces, and we give some qualitative and quantitative results regarding approximate fixed point property and its dense subset property. Furthermore, we formulate and prove results for bounded approximate fixed point sequence and approximate eigenvalues.

Definition 1.1 A subset C of a normed linear space X is

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said to be star shaped if there exists $q \in C$ such that $rx + (1-r)q \in C$ for all $x \in C$ and $0 \le r \le 1$.

Definition 1.2 Let X be a normed linear space and let $T : X \to X$. Then T is said to be homogeneous if $T(\lambda x) = \lambda T(x) \quad \forall \lambda \in M$, where M is a subset of X

Definition 1.3 [9] Let X be a normed linear space and let $T : X \to X$. Then T is demi-closed if for any sequence $\{x_n\}$ weakly convergent to an element x_* with $\{h(x_n)\}$ norm-convergent to an element y_*, then $h(x_*) = y_*$..

Remark 1.4 [14] It is known that if K is a closed convex subset of a uniformly convex Banach space X and $T : K \to X$ a nonexpansive map then $I - T$ is demi-closed.

Definition 1.5 [13] Let K be a subset of a Banach space X. A map $T : K \to X$ is said to be demi-compact at $z \in K$ if for any bounded sequence $\{x_n\} \subseteq K$ such that $(I-T)x_n \to z$ as $n \to \infty$ then there exist a subsequence $\{x_{n_j}\}$ and a point $p \in K$ such that $x_{n_j} \to p$ as $j \to \infty$ and $(I-T)p = z$.

Definition 1.6 [2] Let X be a normed linear space and let $T : X \to X$. Then T is said to be semi compact if, whenever there exist a sequence $\{x_n\} \in X$ with $lim_{n\to\infty} \|Tx_n - x_n\| = 0$, the sequence $\{x_n\}$ has a convergent subsequence.

Definition 1.7 [15] *A mapping* $T : X \to X$ *is a* α*-contraction if there exists* $\alpha \in (0,1)$ *such that*

$$d(Tx, Ty) \le \alpha d(x, y), \ \forall x, y \in X.$$

Remark 1.8 [11] The smallest value of α for which the above inequality holds is said to be Lipschitz constant for T and it is denoted by L. If $L < 1$, T is a contraction and if $L = 1$, T is nonexpansive.

Definition 1.9 [11] Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map. Then $x_0 \in X$ is $\varepsilon-$fixed point for T if $\|Tx_0 - x_0\| < \varepsilon$.

Remark 1.10 [11] In this paper we will denote the set of all $\varepsilon-$ fixed points of T, for a given ε, by :

$$AF(T) = \{x \in X \mid x \text{ is an } \varepsilon - \text{fixed point of } T\}.$$

Definition 1.11 [11] Let (X, d) be a metric space and $T : X \to X$ be a map. Then T has the approximate fixed point property (a.f.p.p) if $\forall \varepsilon > 0$, $AF(T) \ne \varnothing$.

Theorem 1.12 [11] *Let* $(X, \|.\|)$ *be a completely norm space and* $T : X \to X$ *be a map also for all* $x, y \in X$,

$$\|Tx - Ty\| \le c\|x - y\| : \qquad 0 < c < 1$$

then T has an $\varepsilon-$ fixed point in completely norm space. Moreover, if $x, y \in X$ are $\varepsilon-$ fixed points of T, then

$$\|x - y\| \le \frac{2\varepsilon}{1-c}.$$

Remark 1.13 The following result (see [7]) gives conditions under which the existence of fixed points for a given mapping is equivalent to that of approximate fixed points.

Theorem 1.14 Let A be a closed subset of a metric space (X, d) and $T : A \to X$ a compact map. Then T has a fixed point if and only if it has the approximate fixed point property

2. Approximate Fixed Point Property and Its Dense Subset Property

In this section we give some qualitative and quantitative results regarding approximate fixed point property and its dense subset property.

Theorem 2.1 Let $(X, \|.\|)$ be a completely norm space having a.f.p.p and $T : X \to X$ be a nonexpansive map, $x_0 \in X$ and $\varepsilon > 0$. If A be a dense subset of X, then A has a.f.p.p.

Proof: First we claim that,

$$\{x \in X \mid x \text{ is an } \varepsilon - \text{fixed point of } T\} = \\ \{x \in A \mid x \text{ is an } \varepsilon - \text{fixed point of } T\}.$$

We know that

$$\{x \in X \mid x \text{ is an } \varepsilon - \text{fixed point of } T\} \subseteq \\ \{x \in A \mid x \text{ is an } \varepsilon - \text{fixed point of } T\}.$$

Let $y \in X$. Then there exist a sequence $\{y_n\} \in A$ such that $y_n \to y$ and we know that for $x \in A$, $\|Tx - x\| \le \|Ty_n - y_n\| \quad \forall n.$

Taking limit $\|Tx - x\| \le \|Ty - y\|$. This is true for all $y \in X$. Therefore

$$\{x \in A \mid x \text{ is an } \varepsilon - \text{fixed point of } T\} \subseteq \\ \{y \in X \mid y \text{ is an } \varepsilon - \text{fixed point of } T\}.$$

Hence our claim is proved. Now consider any nonexpansive map $T : A \to A$. Given any $x \in X$, there existe a sequence $\{x_n\} \in A$ such that $x_n \to x$ as $n \to \infty$. Then as $T : A \to A$ is continuous it can be Hence our claim is proved. Now consider any nonexpansive map $T : A \to A$. Given any

$x \in X$, there existe a sequence $\{x_n\} \in A$ such that $x_n \to x$ as $n \to \infty$. Then as $T: A \to A$ is continuous it can be extended to X by defining, $Tx = lim_{n\to\infty} Tx_n$. Hence we can consider T as a nonexpansive map on X and then by our claim $\forall \varepsilon > 0, AF(T) \neq \varnothing$, for all $x \in A$, since X has a.f.p.p. Hence A has a.f.p.p. •

Example 2.2 Let C be a closed bounded starshaped subset of a Banach space X with star centre p. If For $n = 1,2,3,\cdots$ define $T_n : C \to C$ by,

$$T_n x = (\frac{n}{n+1})Tx + (\frac{1}{n+1})p, \qquad \forall x \in C$$

such that $T: C \to C$ be a nonexpansive.

Then: $\forall \varepsilon > 0, \quad AF(T) \neq \varnothing$. Since T_n is a contraction on C. Therfore by Theorem 1.12 $AF(T_n) \neq \varnothing, \quad \forall \varepsilon > 0$. Also, by Banach contraction principle it has a unique fixed point x_n in C. Now consider,

$$\|Tx_n - x_n\| = \|Tx_n - T_n x_n\| = \left\|Tx_n - (\frac{n}{n+1})Tx_n - (\frac{1}{n+1})p\right\|$$

$$= (\frac{1}{n+1})\|Tx_n - p\| \qquad \forall n = 1,2,3,\cdots$$

Hence $\|Tx_n - x_n\| < \varepsilon$ for every $\varepsilon > 0$, since C is bounded. Therfore $AF(T) \neq \varnothing, \quad \forall \varepsilon > 0$. •

Example 2.3 Let C be a closed bounded starshaped subset of a Banach space X with star centre p. If For $n = 1,2,3,\cdots$ define $T_n : C \to C$ by,

$$T_n x = (\frac{n}{n+1})Tx + (\frac{1}{n+1})p, \qquad \forall x \in C$$

such that $T: C \to C$ be a nonexpansive, semi-compact map on C. Then $\forall \varepsilon > 0, \quad AF(T) \neq \varnothing$.

By Example 2.2 we have a sequence $\{x_n\}$ in C such that $\|Tx_n - x_n\| < \varepsilon$ for every $\varepsilon > 0$. Also, since T is semi-compact, sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to some $x_* \in C$ and by continuity of T, $Tx_{n_k} \to Tx_*$. Then consider,

$$T_{n_k} x_{n_k} = (\frac{n_k}{n_k+1})Tx_{n_k} + (\frac{1}{n_k+1})p$$

Letting $k \to \infty$, we have $Tx_* = x_*$ and hence x_* is a fixed point of T in C. Now by Theorem 1.14 it follows that $AF(T) \neq \varnothing \quad \forall \varepsilon > 0$. •

Example 2.4 Let C be a closed bounded starshaped subset of a reflexive Banach space X with star centre p. If For $n = 1,2,3,\cdots$ define $T_n : C \to C$ by,

$$T_n x = (\frac{n}{n+1})Tx + (\frac{1}{n+1})p, \qquad \forall x \in C$$

such that $T: C \to C$, $T(C) \subseteq C$ and $I - T$ is demi-closed. Then $\forall \varepsilon > 0, \quad AF(T) \neq \varnothing$.

By Example 2.2 we have a sequence $\{x_n\}$ in C such that $\|Tx_n - x_n\| < \varepsilon$ for every $\varepsilon > 0$. Also, Since X is reflexive Banach space, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to x_*. Since $I - T$ is demi-closed, $(I - T)x_* = 0$, Thus $Tx_* = x_*$. Now by Theorem 1.14 it follows that $AF(T) \neq \varnothing \quad \forall \varepsilon > 0$. ●

Example 2.5 Let C be a closed bounded starshaped subset of a Banach space X with star centre p. If For $n = 1,2,3,\cdots$ define $T_n : C \to C$ by,

$$T_n x = (\frac{n}{n+1})Tx + (\frac{1}{n+1})p, \qquad \forall x \in C$$

such that $T: C \to C$ be a nonexpansive, demi-compact map on C. Then $AF(T)$ is compact and

$$\forall \varepsilon > 0, \quad AF(T) \neq \varnothing.$$

By Example 2.2 we have a sequence $\{x_n\}$ in C such that $\|Tx_n - x_n\| < \varepsilon$ for every $\varepsilon > 0$. Therefore, it has an $\varepsilon-$ fixed point $x_\varepsilon \in C$ and in particular, there exists x_n in C such that

$$\|T_n x_n - x_n\| \leq \frac{1}{n} \quad for \quad n = 1,2,3,\cdots.$$

On the other hand, since

$$\|x_n - Tx_n\| \leq \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\|$$
$$< \frac{1}{n} + \left\|(\frac{n}{n+1})Tx_n + (\frac{1}{n+1})p - Tx_n\right\|$$

$$= (\frac{1}{n}) + (\frac{n}{n+1})\|p - Tx_n\|.$$

Therfore $(I - T)x_n = 0$ as $n \to \infty$. Since $T: C \to C$ is demi-compact, the bounded sequence $\{x_n\}$ in C has a convergent subsequence $\{x_{n_j}\}$ which converges to a point say $\{x_0\}$ in C as $j \to \infty$ and $(I-T)x_0 = 0$. Hence $Tx_* = x_*$. Now by Theorem 1.14 it follows that $AF(T) \neq \emptyset \quad \forall \varepsilon > 0$. Furthermore, $AF(T)$ is also compact sinc T is demi-compact on C. •

3. Bounded Approximate Fixed Point Sequence Property and Approximate Eigenvalues

In this section we formulate and prove results for bounded approximate fixed point sequence and approximate eigenvalues.

Definition 3.1 *Let C be a closed bounded starshaped subset of a Banach space X with star centre p, $\varepsilon > 0$ and $T: C \to C$ be a map. Then $\{x_n\} \in C$ is bounded $\varepsilon-$fixed point sequence for T if $\|Tx_n - x_n\| < \varepsilon$ for $n = 1,2,3,\ldots$.*

Remark 3.2 In this paper we will denote the bounded $\varepsilon-$ fixed point sequence of T, for a given ε, by :

$$BAFS(T) = \{\{x_n\} \in C : x_n \text{ is a bounded } \varepsilon-\text{fixed point sequence of } T\}.$$

Definition 3.3 Let C be a closed bounded starshaped subset of a Banach space X with star centre p, $\varepsilon > 0$ and $T: C \to C$ be a map. Then T has the bounded approximate fixed point sequence property (b.a.f.p.s.p) if

$$\forall \varepsilon > 0, BAFS(T) \neq \emptyset.$$

Definition 3.4 *Let K be a subset of a completely norm space (X,d) and $T: K \to K$ be a map. Then $\lambda_\varepsilon \in K$ is $\varepsilon-$eigenvalue for T if there exists a sequence $\{x_n\} \in X$ such that $\|x_n\| = 1$ for each n and $\|Tx_n - \lambda_\varepsilon x_n\| < \varepsilon$.*

Remark 3.5 In this paper we will denote the $\varepsilon-$eigenvalue of T, for a given ε, by :

$$AEIG(T) = \{\lambda_\varepsilon \in K : \lambda_\varepsilon \textit{ is an } \varepsilon-\textit{eigenvalue of } T\}.$$

Theorem 3.6 Let C be a closed bounded starshaped subset of a Banach space X with star centre p. If For $n = 1,2,3,\cdots$ define $T_n : C \to C$ by,

$$T_n x = \left(\frac{n}{n+1}\right)Tx + \left(\frac{1}{n+1}\right)p, \qquad \forall x \in C$$

such that $T: C \to C$ be a nonexpansive, Then: $\forall \varepsilon > 0, \quad BAFS(T) \neq \emptyset.$

Proof: Since T_n is a contraction on C. Therefore by Banach contraction principle it has a unique fixed point x_n in C. Now consider,

$$\|Tx_n - x_n\| = \|Tx_n - T_n x_n\|$$

$$= \left\|Tx_n - \left(\frac{n}{n+1}\right)Tx_n - \left(\frac{1}{n+1}\right)p\right\|$$

$$= \left(\frac{1}{n+1}\right)\|Tx_n - p\| \qquad \forall n = 1,2,3,\cdots$$

Hence $\|Tx_n - x_n\| < \varepsilon$ for every $\varepsilon > 0$, since C is bounded. Therfore $BAFS(T) \neq \emptyset, \quad \forall \varepsilon > 0$. •

Theorem 3.7 Let A be a subset of a normed linear space X such that $\lambda_\varepsilon A \subseteq A, \forall \lambda_\varepsilon > 0$ *and* $0 \notin A$. Also suppose that $BAFS(T) \neq \emptyset, \forall \varepsilon > 0$ and $T: A \to A$ is a nonexpansive homogeneous mapping. If $\lambda_\varepsilon \geq 1, \lambda_\varepsilon$ is an approximate eigenvalue of T.

Proof: Let $\lambda_\varepsilon \geq 1$ Consider a map $T_{\lambda_\varepsilon} : A \to A$ defined by

$$T_{\lambda_\varepsilon}(x) := \frac{1}{\lambda_\varepsilon} T(x) \quad \textit{for} \quad x \in A.$$

Then T_{λ_ε} is a homogeneous nonexpansive map on A. Since $\forall \varepsilon > 0, \ BAFS(T) \neq \emptyset$, there exist a bounded sequence $\{x_n\}$ in A such that $\|T_{\lambda_\varepsilon} x_n - x_n\| < \varepsilon$. By our assumption $y_n = \frac{x_n}{\|x_n\|}$ exists and belongs to A for every n. Now, Consider

$$\|T_{\lambda_\varepsilon} y_n - y_n\| = \left\|T_{\lambda_\varepsilon}\left(\frac{x_n}{\|x_n\|}\right) - \left(\frac{x_n}{\|x_n\|}\right)\right\|$$

$$= \frac{1}{\|x_n\|}\|T_{\lambda_\varepsilon} x_n - x_n\|$$

$$< \left(\frac{1}{\|x_n\|}\right)\varepsilon.$$

Hence, $\left\|\frac{1}{\lambda_\varepsilon} Ty_n - y_n\right\| < \varepsilon$, i.e., $\|Ty_n - \lambda_\varepsilon y_n\| < \varepsilon$.

Therefore there exists a sequence $\{y_n\}$ in A such that $\|y_n\| = 1$ and $\|Ty_n - \lambda_\varepsilon y_n\| < \varepsilon$. Hence λ_ε is an $\varepsilon-$

eigenvalue of T. •

Theorem 3.8 Let A be a subset of a normed linear space X such that $\lambda_\varepsilon A \subseteq A, \forall \lambda_\varepsilon > 0$ *and* $0 \notin A$. Also suppose that $BAFS(T) \neq \emptyset, \forall \varepsilon > 0$. Then for any homogeneous Lipschitzian map $T : A \to A$, the Lipschitzian constant L is an approximate eigen value of T.

Proof**:** Let $L \geq 1$ define $T_L x = \frac{1}{L} Tx \quad for \quad x \in A.$

Then T_L is a homogeneous nonexpansive map on A. Hence, as in Theorem 3.7 L is an $\varepsilon -$eigenvalue of T.

4. Conclusions

Nowadays, fixed points and approximate fixed points in the both metric and normed spaces play an important role in different areas of mathematics and its applications, particularly in mathematics, physics, differential equation, game theory, and dynamic programming.

In this work, we introduced the bounded approximate fixed points property for self-maps on Banach space and gave results about approximate eigenvalue and bounded approximate fixed point on normed spaces. Also, we proved several approximate eigenvalue and bounded approximate fixed point theorems on normed spaces .

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