Systems Science and Applied Mathematics
Vol. 2, No. 1, 2017, pp. 1-7
http://www.aiscience.org/journal/ssam

(ρ,b)-Quasiinvexity and Efficiency Conditions in Matrix Variational Problems

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Abstract

The main goal of this paper is to formulate and prove necessary and sufficient conditions of efficiency for a class of matrix variational problems (*MVP*). Under (*ρ,b*)-quasiinvexity assumptions, we establish sufficient efficiency conditions for a feasible solution in (*MVP*). The method of investigation used in this work is based on employing of several adequate variational calculus techniques in the study of some vector variational problems. Our results extend, unify and improve several theorems in the current literature.

Keywords

(Normal) Efficient Solution, (*ρ,b*)-Quasiinvexity, Matrix Variational Problem

Received: March 9, 2017 / Accepted: April 7, 2017 / Published online: June 15, 2017

1. Introduction and Preliminaries

Valentine [1] has established in 1937 the first result on the necessity of optimal solutions for scalar variational problems. Motivated by its applications in engineering problems and mechanics, we initiate an optimization theory for matrix variational problems, considering the problem (*MVP*) of minimization for matrices of simple integral functionals. Certainly, the applicability area will be much larger if we consider path-independent curvilinear integral functionals (well known as mechanical work) instead of simple integral functionals, but this subject will be explored and studied in a future work. In this work, we introduce the notion of *(normal) efficient solution* (see also [2], [3], [4]) for a class of matrix variational problems (*MVP*). We use a method of investigation based on employing of several adequate variational calculus techniques in the study of some vector variational problems (see (*VVP*)). This approach allows us to state and prove efficiency conditions for matrix variational problems (see (*MVP*)) subject to ordinary differential equations (ODEs) and/or ordinary differential inequations (ODIs) constraints.

Let $I:=[t_0,t_1]\subseteq R$ be a real interval, $f=\left(f_{\alpha\beta}\right):I\times R^n\times R^n\to R^{pq}$, $\alpha=\overline{1,p}$, $\beta=\overline{1,q}$, a C^2-class function, that determines the following $p\times q$ matrix

$$\begin{pmatrix} f_{11}(t,x(t),\dot{x}(t)) & \cdots & f_{1q}(t,x(t),\dot{x}(t)) \\ \vdots & \ddots & \vdots \\ f_{p1}(t,x(t),\dot{x}(t)) & \cdots & f_{pq}(t,x(t),\dot{x}(t)) \end{pmatrix} \tag{1}$$

and let $g=(g_1,\dots,g_m):I\times R^n\times R^n\to R^m$, $h=(h_1,\dots,h_r):I\times R^n\times R^n\to R^r$ be twice differentiable functions.

The C^2-class Lagrangians, $f_{\alpha\beta}(t,x(t),\dot{x}(t))$, $\alpha=\overline{1,p}$, $\beta=\overline{1,q}$, generate the following functionals

$$F_{\alpha\beta}(x(\cdot)):=\int_{t_0}^{t_1} f_{\alpha\beta}\,(t,x(t),\dot{x}(t))dt, \tag{2}$$

where $x:[t_0,t_1]\to R^n$ and $\dot{x}(t):=\frac{d}{dt}x(t)$.

Let $C^\infty([t_0,t_1],R^n)$ be the space of all functions $x:[t_0,t_1]\to R^n$ of C^∞-class, endowed with the norm

$$\|x\|:=\|x\|_\infty+\|\dot{x}\|_\infty. \tag{3}$$

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In our subsequent theory, we shall consider the following relations between two vectors $u = (u_1, \dots, u_s)$, $v = (v_1, \dots, v_s)$ in R^s:

$$u = v \Leftrightarrow u_i = v_i, \quad u \le v \Leftrightarrow u_i \le v_i \tag{4}$$

$$u < v \Leftrightarrow u_i < v_i, \quad u \preccurlyeq v \Leftrightarrow u \le v,\ u \ne v, \quad i = \overline{1,s}.$$

Let us denote by

$$\mathrm{F}(I) := \{x \in C^{\infty}(I, R^n) | x(t_0) = x_0, x(t_1) = x_1, g(t, x, \dot{x}) \le 0, h(t, x, \dot{x}) = 0, t \in I\} \tag{5}$$

the *set of all feasible solutions (domain)* for the following matrix variational problem (MVP),

$$\min_{x(\cdot)} \begin{pmatrix} \int_{t_0}^{t_1} f_{11}(t, x(t), \dot{x}(t))dt & \cdots & \int_{t_0}^{t_1} f_{1q}(t, x(t), \dot{x}(t))dt \\ \vdots & \ddots & \vdots \\ \int_{t_0}^{t_1} f_{p1}(t, x(t), \dot{x}(t))dt & \cdots & \int_{t_0}^{t_1} f_{pq}(t, x(t), \dot{x}(t))dt \end{pmatrix} \tag{6}$$

subject to $x(\cdot) \in \mathrm{F}(I)$.

In this paper, we are looking for necessary and sufficient efficiency conditions in the matrix variational problem (MVP).

In order to analyze the multiobjective variational problem (MVP), we start with the case of a single functional. Let us consider the following scalar variational problem (SVP),

$$\min_{x(\cdot)} \left\{ I(x(\cdot)) = \int_{t_0}^{t_1} X(t, x(t), \dot{x}(t))dt \right\} \tag{7}$$

subject to

$$x(t_0) = x_0, \quad x(t_1) = x_1, \tag{8}$$

$$g(t, x(t), \dot{x}(t)) \le 0, \quad h(t, x(t), \dot{x}(t)) = 0, \quad (\forall) t \in I. \tag{9}$$

Define the auxiliary Lagrange function L as (see summation upon the repeated indices!)

$$L(t, x(t), \dot{x}(t), p(t), q(t), \lambda) := \lambda X(t, x(t), \dot{x}(t)) + p^a(t) g_a(t, x(t), \dot{x}(t)) \tag{10}$$

$$+ q^{\varsigma}(t) h_{\varsigma}(t, x(t), \dot{x}(t)).$$

The necessary conditions for the optimality of a feasible solution x^0 in the problem (SVP) have been given for the first time by Valentine (see [1]) in the following form:

$$\frac{\partial L}{\partial x}(t, x^0(t), \dot{x}^0(t), p(t), q(t), \lambda) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x^0(t), \dot{x}^0(t), p(t), q(t), \lambda) = 0 \tag{11}$$

(Euler-Lagrange ODEs)

$$p(t) g(t, x^0(t), \dot{x}^0(t)) = 0, \quad p(t) \ge 0, \quad (\forall) t \in I, \tag{12}$$

or, equivalently:

Theorem 1.1 (Valentine's necessary conditions, [1]) *Assume that x^0, a feasible solution of the problem (SVP), is an optimal solution and X, g, h are functions of C^2-class. Then there exist a scalar λ and the piecewise smooth functions $p(t)$ and $q(t)$, satisfying the following conditions*

$$\lambda \frac{\partial X}{\partial x}(t, x^0, \dot{x}^0) + p^a(t) \frac{\partial g_a}{\partial x}(t, x^0, \dot{x}^0) + q^{\varsigma}(t) \frac{\partial h_{\varsigma}}{\partial x}(t, x^0, \dot{x}^0) \tag{13}$$

$$= \frac{d}{dt}\left\{ \lambda \frac{\partial X}{\partial \dot{x}}(t, x^0, \dot{x}^0) + p^a(t) \frac{\partial g_a}{\partial \dot{x}}(t, x^0, \dot{x}^0) + q^{\varsigma}(t) \frac{\partial h_{\varsigma}}{\partial \dot{x}}(t, x^0, \dot{x}^0) \right\}$$

(Euler-Lagrange ODEs)

$$p(t) g\big(t, x^0(t), \dot{x}^0(t)\big) = 0, \quad p(t) \ge 0, \quad (\forall) t \in I. \tag{14}$$

Definition 1.1 If $\lambda \ne 0$ in the previous framework, the optimal solution $x^0(\cdot)$ of the problem (SVP) is called *normal optimal solution*.

Without a loss of generality, we can assume that $\lambda = 1$.

Consider now the following vector variational problem (VVP),

$$\min_{x(\cdot)} \left(\int_{t_0}^{t_1} f_1(t, x(t), \dot{x}(t))dt, \dots, \int_{t_0}^{t_1} f_p(t, x(t), \dot{x}(t))dt \right) \tag{15}$$

subject to $x(\cdot) \in \mathrm{F}(I)$.

Further, we formulate the necessary and sufficient conditions (efficiency conditions) for the optimality of the vector optimization problem (VVP) in the domain $\mathrm{F}(I)$.

Definition 1.2 ([5]) A feasible solution $x^0 \in \mathrm{F}(I)$ is called *efficient solution* for (VVP) if there exists no other feasible solution $x \in \mathrm{F}(I)$ such that

$$\int_{t_0}^{t_1} f(t, x, \dot{x})dt \preccurlyeq \int_{t_0}^{t_1} f(t, x^0, \dot{x}^0)dt, \tag{16}$$

where

$$\int_{t_0}^{t_1} f(t, x, \dot{x})dt := \left(\int_{t_0}^{t_1} f_1(t, x, \dot{x})dt, \dots, \int_{t_0}^{t_1} f_p(t, x, \dot{x})dt \right). \tag{17}$$

Theorem 1.2 (Normal necessary efficiency conditions for (VVP), [3]) *Let $x^0(\cdot) \in \mathrm{F}(I)$ be a normal efficient solution of the problem (VVP). Then there exist $\lambda \in R^p$ and the piecewise smooth functions $p: I \to R^m$ and $q: I \to R^r$ that fulfil the following conditions:*

$$\sum_{j=1}^{p} \lambda_j \frac{\partial f_j}{\partial x}(t, x^0, \dot{x}^0) + p(t) \frac{\partial g}{\partial x}(t, x^0, \dot{x}^0) + q(t) \frac{\partial h}{\partial x}(t, x^0, \dot{x}^0) \tag{18}$$

$$= \frac{d}{dt}\left\{ \sum_{j=1}^{p} \lambda_j \frac{\partial f_j}{\partial \dot{x}}(t, x^0, \dot{x}^0) + p(t) \frac{\partial g}{\partial \dot{x}}(t, x^0, \dot{x}^0) + q(t) \frac{\partial h}{\partial \dot{x}}(t, x^0, \dot{x}^0) \right\},$$

$$p(t) g(t, x^0, \dot{x}^0) = 0, \quad p(t) \ge 0, \quad (\forall) t \in I, \lambda \ge 0, \tag{19}$$

$$e^t\lambda = 1, \quad e^t = (1,1,\dots,1) \in R^p. \tag{20}$$

In order to get sufficient efficiency conditions for (VVP), we shall introduce the notion of (ρ, b)-quasiinvexity, used in recent works for the study of some multiobjective optimization problems (see [2], [3], [4], [6], [7]). Let us consider $\rho \in R$, $b: C^\infty([t_0,t_1],R^n) \times C^\infty([t_0,t_1],R^n) \to [0,\infty)$ a functional, and $f: R \times R^n \times R^n \to R$ a real function that determines the following simple integral functional

$$A(x(\cdot)) = \int_{t_0}^{t_1} f\,(t,x(t),\dot{x}(t))dt. \tag{21}$$

Definition 1.3 We say that functional $A(x)$ is *[strictly] (ρ,b)-quasiinvex at x^0* if there exist the vector functions $\eta = (\eta_1,\dots,\eta_n)$, vanishing at the point $(t,x^0,\dot{x}^0)$, and $\theta: C^\infty([t_0,t_1],R^n) \times C^\infty([t_0,t_1],R^n) \to R^n$ such that, for any x $[x \neq x^0]$, the following implication holds

$$A(x) \le A(x^0) \Rightarrow b(x,x^0)\int_{t_0}^{t_1} \eta\,(t,x,\dot{x}^0)\frac{\partial f}{\partial x}(t,x^0,\dot{x}^0)dt \tag{22}$$

$$+b(x,x^0)\int_{t_0}^{t_1}\frac{d\eta}{dt}(t,x,\dot{x}^0)\frac{\partial f}{\partial \dot{x}}(t,x^0,\dot{x}^0)dt\ [<]$$
$$\le -\rho b(x,x^0)\parallel \theta(x,x^0)\parallel^2.$$

Theorem 1.3 (Sufficient efficiency conditions for (VVP), [3]) *Let* $x^0(\cdot) \in \mathrm{F}(I)$, $\lambda \in R^p$, $p: I \to R^m$, $q: I \to R^r$ *that fulfil the conditions in Theorem* 1.2. *Also, consider that the following statements hold:*

a) the functionals $\int_{t_0}^{t_1} f_l\,(t,x(t),\dot{x}(t))dt$, $l \in \{1,\dots,p\}$, are (ρ_l^1, b)-quasiinvex at $x^0(\cdot)$ with respect to η, θ;

b) $\int_{t_0}^{t_1} p\,(t)g(t,x(t),\dot{x}(t))dt$ is (ρ^2,b)-quasiinvex at $x^0(\cdot)$ with respect to η, θ;

c) $\int_{t_0}^{t_1} q\,(t)h(t,x(t),\dot{x}(t))dt$ is (ρ^3,b)-quasiinvex at $x^0(\cdot)$ with respect to η, θ;

d) one of the functionals $\int_{t_0}^{t_1} f_l\,(t,x(t),\dot{x}(t))dt$, $l \in \{1,\dots,p\}$, $\int_{t_0}^{t_1} p\,(t)g(t,x(t),\dot{x}(t))dt$, $\int_{t_0}^{t_1} q\,(t)h(t,x(t),\dot{x}(t))dt$ is strictly (ρ,b)-quasiinvex at $x^0(\cdot)$ with respect to η, θ ; ($\rho = \rho_l^1$, ρ^2 or ρ^3 , respectively)

e) $\sum_{l=1}^{p}\lambda_l\rho_l^1 + \rho^2 + \rho^3 \ge 0$ $\quad(\rho_l^1,\ \rho^2,\ \rho^3 \in R)$.

Then $x^0(\cdot)$ is an efficient solution for (VVP).

2. Main Results

The main goal of this section is to provide necessary and sufficient conditions of efficiency for the following matrix variational problem (*MVP*):

$$\min_{x(\cdot)}\begin{pmatrix}\int_{t_0}^{t_1} f_{11}(t,x(t),\dot{x}(t))dt & \cdots & \int_{t_0}^{t_1} f_{1q}(t,x(t),\dot{x}(t))dt\\ \vdots & \ddots & \vdots\\ \int_{t_0}^{t_1} f_{p1}(t,x(t),\dot{x}(t))dt & \cdots & \int_{t_0}^{t_1} f_{pq}(t,x(t),\dot{x}(t))dt\end{pmatrix} \tag{23}$$

subject to $x(\cdot) \in \mathrm{F}(I)$.

Definition 2.1 A feasible solution $x^0(\cdot) \in \mathrm{F}(I)$ is called *efficient solution* for (MVP) if there exists no feasible solution $x(\cdot) \in \mathrm{F}(I)$ such that

$$\left(\int_{t_0}^{t_1} f_{\alpha\beta}\,(t,x(t),\dot{x}(t))dt\right) \le \left(\int_{t_0}^{t_1} f_{\alpha\beta}\,(t,x^0(t),\dot{x}^0(t))dt\right), \quad \alpha = \overline{1,p},\ \beta = \overline{1,q} \tag{24}$$

and

$$\left(\int_{t_0}^{t_1} f_{\alpha\beta}\,(t,x(t),\dot{x}(t))dt\right) \neq \left(\int_{t_0}^{t_1} f_{\alpha\beta}\,(t,x^0(t),\dot{x}^0(t))dt\right), \quad \alpha = \overline{1,p},\ \beta = \overline{1,q}, \tag{25}$$

where

$$\left(\int_{t_0}^{t_1} f_{\alpha\beta}\,(\cdot,\cdot,\cdot)dt\right), \quad \alpha = \overline{1,p},\ \beta = \overline{1,q} \tag{26}$$

represents the following vector in R^{pq}

$$\left(\int_{t_0}^{t_1} f_{11}\,(\cdot,\cdot,\cdot)dt,\dots,\int_{t_0}^{t_1} f_{1q}\,(\cdot,\cdot,\cdot)dt,\dots,\int_{t_0}^{t_1} f_{p1}\,(\cdot,\cdot,\cdot)dt,\dots,\int_{t_0}^{t_1} f_{pq}\,(\cdot,\cdot,\cdot)dt\right). \tag{27}$$

Remark that, taking into account Definition 2.1, we can transform our matrix minimization problem (*MVP*) into a vector minimization problem. In this direction, let establish the following

Lemma 2.1 ([8]) *The feasible solution $x^0(\cdot) \in F(I)$ is an efficient solution of the problem (MVP) if and only if it is an optimal solution of each scalar problem*

$$\min_{x(\cdot)} \int_{t_0}^{t_1} f_{ls}\,(t,x(t),\dot{x}(t))dt \tag{28}$$

subject to

$$x(t_0) = x_0, \quad x(t_1) = x_1 \tag{29}$$

$$g(t,x(t),\dot{x}(t)) \le 0, \quad h(t,x(t),\dot{x}(t)) = 0, \quad (\forall)t \in I \tag{30}$$

$$\int_{t_0}^{t_1} f_{jk}\,(t,x(t),\dot{x}(t))dt \le \int_{t_0}^{t_1} f_{jk}\,(t,x^0(t),\dot{x}^0(t))dt \tag{31}$$

$$j = \overline{1,p}, \quad j \neq l, \quad k = \overline{1,q}, \quad k \neq s. \tag{32}$$

Proof. "$\Longrightarrow$" We prove the direct implication. Consider $x^0(\cdot) \in \mathrm{F}(I)$ an efficient solution of the problem (MVP). Suppose there exist $\alpha = \overline{1,p}$ and $\beta = \overline{1,q}$ such that $x^0(\cdot) \in \mathrm{F}(I)$ is not an optimal solution of the scalar problem $P_{\alpha\beta}(x^0)$. Consequently, there exists a function $y(\cdot) \in \mathrm{F}(I)$ such that

$$\int_{t_0}^{t_1} f_{jk}\,(t,y(t),\dot{y}(t))dt \le \int_{t_0}^{t_1} f_{jk}\,(t,x^0(t),\dot{x}^0(t))dt \tag{33}$$

$$j=\overline{1,p}, \quad j\neq\alpha, \quad k=\overline{1,q}, \quad k\neq\beta \tag{34}$$

and

$$\int_{t_0}^{t_1} f_{\alpha\beta}\,(t,y(t),\dot{y}(t))dt < \int_{t_0}^{t_1} f_{\alpha\beta}\,(t,x^0(t),\dot{x}^0(t))dt. \tag{35}$$

The foregoing relations contradict the efficiency of the function $x^0(\cdot)\in \mathrm{F}(I)$ in (MVP).

"⟸" Let $x^0(\cdot)\in \mathrm{F}(I)$ be an optimal solution of each scalar problem $P_{ls}(x^0)$, $l=\overline{1,p}$, $s=\overline{1,q}$. Assume that $x^0(\cdot)\in \mathrm{F}(I)$ is not an efficient solution of the problem (MVP). Therefore, there exists a function $y(\cdot)\in \mathrm{F}(I)$ such that

$$\int_{t_0}^{t_1} f_{jk}\,(t,y(t),\dot{y}(t))dt \leq \int_{t_0}^{t_1} f_{jk}\,(t,x^0(t),\dot{x}^0(t))dt, \quad j=\overline{1,p}, \quad k=\overline{1,q} \tag{36}$$

and there exist $\alpha=\overline{1,p}$ and $\beta=\overline{1,q}$ such that

$$\int_{t_0}^{t_1} f_{\alpha\beta}\,(t,y(t),\dot{y}(t))dt < \int_{t_0}^{t_1} f_{\alpha\beta}\,(t,x^0(t),\dot{x}^0(t))dt. \tag{37}$$

We obtain a contradiction of our assumption: the function $x^0(\cdot)\in \mathrm{F}(I)$ minimizes the functional $\int_{t_0}^{t_1} f_{\alpha\beta}\,(t,x(t),\dot{x}(t))dt$ on the set of all feasible solutions of problem $P_{\alpha\beta}(x^0)$. The proof is complete.

Lemma 2.2 *Let $l=\overline{1,p}$ and $s=\overline{1,q}$ be fixed. If $x^0(\cdot)\in F(I)$ is an optimal solution of the scalar problem $P_{ls}(x^0)$, then there exist the real scalars $\lambda_{ls;jk}\geq 0$ and the piecewise smooth functions $p_{ls}(t)$ and $q_{ls}(t)$ such that*

$$\sum_{j=\overline{1,p};\ k=\overline{1,q}} \lambda_{ls;jk}\frac{\partial f_{jk}}{\partial x}(t,x^0,\dot{x}^0)+p_{ls}(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0)+q_{ls}(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0) \tag{38}$$

$$=\frac{d}{dt}\left\{\sum_{j=\overline{1,p};\ k=\overline{1,q}} \lambda_{ls;jk}\frac{\partial f_{jk}}{\partial \dot{x}}(t,x^0,\dot{x}^0)+p_{ls}(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0)+q_{ls}(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right\}$$

(Euler-Lagrange ODEs)

$$p_{ls}(t)g(t,x^0,\dot{x}^0)=0, \quad p_{ls}(t)\geq 0, \quad (\forall)t\in I. \tag{39}$$

Proof. Consider the following Lagrangian

$$V_{ls}(t,x,\dot{x},p_{ls},q_{ls},a_{jk}) \tag{40}$$

$$:=\gamma_{ls}\left\{f_{ls}(t,x,\dot{x})+\sum_{j=\overline{1,p},\ j\neq l;\ k=\overline{1,q},\ k\neq s}\lambda_{ls;jk}\left[f_{jk}(t,x,\dot{x})-R_{jk}^0+\phi_{jk}(t,x,\dot{x})\right]\right\}$$

$$+p_{ls}(t)g(t,x,\dot{x})+q_{ls}(t)h(t,x,\dot{x})-\sum_{j=\overline{1,p},\ j\neq l;\ k=\overline{1,q},\ k\neq s} a_{jk}(t)\phi_{jk}(t,x,\dot{x}),$$

where $\gamma_{ls}\in R$, $\gamma_{ls}\geq 0$, and $p_{ls}:I\to R^m$, $q_{ls}:I\to R^r$, $a_{jk}:I\to R$, $a_{jk}\geq 0$, $j=\overline{1,p}$, $j\neq l$, $k=\overline{1,q}$, $k\neq s$, are piecewise smooth functions.

Also, $R_{ls}^0:=\int_{t_0}^{t_1} f_{ls}\,(t,x^0(t),\dot{x}^0(t))dt = \min_{x(\cdot)}\int_{t_0}^{t_1} f_{ls}\,(t,x(t),\dot{x}(t))dt$, $l=\overline{1,p}$, $s=\overline{1,q}$, and $\phi_{jk}:I\times R^n\times R^n\to R$, $\phi_{jk}(t,x,\dot{x})\geq 0$, $j=\overline{1,p}$, $j\neq l$, $k=\overline{1,q}$, $k\neq s$, are twice differentiable functions defined as

$$G_{jk}(x(t)):=\int_{t_0}^{t_1}\left[f_{jk}(t,x(t),\dot{x}(t))-R_{jk}^0+\phi_{jk}(t,x(t),\dot{x}(t))\right]dt=0. \tag{41}$$

By hypothesis, the function x^0 is an optimal solution of the scalar problem $P_{ls}(x^0)$. Consequently, the following Valentine's necessary condition hold

$$\gamma_{ls}\frac{\partial f_{ls}}{\partial x}(t,x^0,\dot{x}^0)+\sum_{j=\overline{1,p},\ j\neq l;\ k=\overline{1,q},\ k\neq s}\gamma_{ls}\lambda_{ls;jk}\frac{\partial f_{jk}}{\partial x}(t,x^0,\dot{x}^0) \tag{42}$$

$$+\sum_{j=\overline{1,p},\ j\neq l;\ k=\overline{1,q},\ k\neq s}\left[\gamma_{ls}\lambda_{ls;jk}-a_{jk}(t)\right]\frac{\partial \phi_{jk}}{\partial x}(t,x^0,\dot{x}^0)$$

$$+p_{ls}(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0)+q_{ls}(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0)$$

$$=\frac{d}{dt}\left\{\gamma_{ls}\frac{\partial f_{ls}}{\partial \dot{x}}(t,x^0,\dot{x})+\sum_{j=\overline{1,p},\ j\neq l;\ k=\overline{1,q},\ k\neq s}\gamma_{ls}\lambda_{ls;jk}\frac{\partial f_{jk}}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right\}$$

$$+\frac{d}{dt}\left\{\sum_{j=\overline{1,p},\ j\neq l;\ k=\overline{1,q},\ k\neq s}\left[\gamma_{ls}\lambda_{ls;jk}-a_{jk}(t)\right]\frac{\partial \phi_{jk}}{\partial \dot{x}}(t,x^0,\dot{x}^0)+p_{ls}(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right\}$$

$$+\frac{d}{dt}\left\{q_{ls}(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right\}.$$

We impose the following conditions: $\gamma_{ls}\lambda_{ls;jk}-a_{jk}(t)=0$, $j=\overline{1,p}$, $j\neq l$, $k=\overline{1,q}$, $k\neq s$, for any $t\in I$; $\gamma_{ls}=\lambda_{ls,ls}\geq 0$, $\lambda_{ls;jk}=\gamma_{ls}\lambda_{ls;jk}\geq 0$, $j=\overline{1,p}$, $j\neq l$, $k=\overline{1,q}$, $k\neq s$. Rewriting (42), we find

$$\lambda_{ls,ls}\frac{\partial f_{ls}}{\partial x}(t,x^0,\dot{x})+\sum_{j=\overline{1,p},\ j\neq l;\ k=\overline{1,q},\ k\neq s}\lambda_{ls,jk}\frac{\partial f_{jk}}{\partial x}(t,x^0,\dot{x}^0) \tag{43}$$

$$+p_{ls}(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x})+q_{ls}(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0)$$

$$= \frac{d}{dt}\left\{\lambda_{ls,ls}\frac{\partial f_{ls}}{\partial \dot{x}}(t,x^0,\dot{x}^0) + \sum_{j=\overline{1,p},\ j\neq l;\ k=\overline{1,q},\ k\neq s}\lambda_{ls;jk}\frac{\partial f_{jk}}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right\}$$

$$+ \frac{d}{dt}\left\{p_{ls}(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0) + q_{ls}(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right\}$$

and the proof is complete.

Definition 2.2 The feasible solution $x^0(\cdot) \in \mathrm{F}(I)$ is a *normal efficient solution* of the problem (MVP) if it is a normal optimal solution for at least one of the problems $P_{ls}(x^0)$, $l = \overline{1,p}$, $s = \overline{1,q}$.

At this moment, we are able to provide the first main result of our paper, namely, necessary efficiency conditions for (MVP).

Theorem 2.1 ([Normal] necessary efficiency conditions for (MVP)) *Suppose that $x^0(\cdot) \in \mathrm{F}(I)$ is a [normal] efficient solution of the matrix variational problem (MVP). Then there exist $\lambda \in R^{pq}$ and the piecewise smooth functions $p: I \to R^m$ and $q: I \to R^r$ that fulfil the following conditions:*

$$\sum_{j=\overline{1,p};\ k=\overline{1,q}}\lambda_{jk}\frac{\partial f_{jk}}{\partial x}(t,x^0,\dot{x}^0) + p(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0) + q(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0) \quad (44)$$

$$= \frac{d}{dt}\left\{\sum_{j=\overline{1,p};\ k=\overline{1,q}}\lambda_{jk}\frac{\partial f_{jk}}{\partial \dot{x}}(t,x^0,\dot{x}^0) + p(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0) + q(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right\}$$

$$p(t)g(t,x^0,\dot{x}^0) = 0, \quad p(t) \geq 0, \quad (\forall)t \in I, \lambda \geq 0, \quad (45)$$

$$e^t\lambda = 1, \quad e^t = (1,1,\ldots,1) \in R^{pq}. \quad (46)$$

Proof. Taking into account Lemma 2.1 , we have that $x^0(\cdot) \in \mathrm{F}(I)$ is an optimal solution of each problems $P_{ls}(x^0)$, $l = \overline{1,p}$, $s = \overline{1,q}$. If $x^0(\cdot) \in \mathrm{F}(I)$ is [normal] optimal solution in $P_{ls}(x^0)$, $l \in \{1,\ldots,p\}$, $s \in \{1,\ldots,q\}$ fixed, then the relations in Lemma 2.2 are true $[\lambda_{ls;ls} = 1]$. We make summation over $l = \overline{1,p}$, $s = \overline{1,q}$ of all relations in Lemma 2.2 and set

$$\sum_{l=\overline{1,p};\ s=\overline{1,q}}\lambda_{ls;jk} = \tilde{\lambda}_{jk}, \quad \sum_{l=\overline{1,p};\ s=\overline{1,q}}p_{ls}(t) = \tilde{p}(t), \quad \sum_{l=\overline{1,p};\ s=\overline{1,q}}q_{ls}(t) = \tilde{q}(t). \quad (47)$$

We get

$$\sum_{j=\overline{1,p};\ k=\overline{1,q}}\tilde{\lambda}_{jk}\frac{\partial f_{jk}}{\partial x}(t,x^0,\dot{x}^0) + \tilde{p}(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0) + \tilde{q}(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0) \quad (48)$$

$$= \frac{d}{dt}\left\{\sum_{j=\overline{1,p};\ k=\overline{1,q}}\tilde{\lambda}_{jk}\frac{\partial f_{jk}}{\partial \dot{x}}(t,x^0,\dot{x}^0) + \tilde{p}(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0) + \tilde{q}(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right\}$$

$$\tilde{p}(t)g(t,x^0,\dot{x}^0) = 0, \quad \tilde{p}(t) \geq 0, \quad \tilde{\lambda}_{jk} \geq 0, \quad (\forall)t \in I, \quad [\tilde{\lambda}_{jk} \geq 1] \quad (49)$$

and by dividing with $S = \sum_{j=\overline{1,p};\ k=\overline{1,q}}\tilde{\lambda}_{jk} \geq 1$ and denoting $\lambda_{jk} = \tilde{\lambda}_{jk}/S$, $p(t) = \tilde{p}(t)/S$, $q(t) = \tilde{q}(t)/S$, we obtain

$$\sum_{j=\overline{1,p};\ k=\overline{1,q}}\lambda_{jk}\frac{\partial f_{jk}}{\partial x}(t,x^0,\dot{x}^0) + p(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0) + q(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0) \quad (50)$$

$$= \frac{d}{dt}\left\{\sum_{j=\overline{1,p};\ k=\overline{1,q}}\lambda_{jk}\frac{\partial f_{jk}}{\partial \dot{x}}(t,x^0,\dot{x}^0) + p(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}) + q(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right\}$$

$$p(t)g(t,x^0,\dot{x}^0) = 0, \quad p(t) \geq 0, \quad (\forall)t \in I \quad (51)$$

$$\lambda \geq 0, \quad e^t\lambda = 1, \quad e^t = (1,1,\ldots,1) \in R^{pq} \quad (52)$$

and the proof is complete.

Let establish the second main result of this section: sufficient efficiency conditions for the matrix variational problem (*MVP*).

Theorem 2.2 (Sufficient efficiency conditions for (MVP)) *Let $x^0(\cdot) \in \mathrm{F}(I)$ and $\lambda \in R^{pq}$, $p: I \to R^m$, $q: I \to R^r$ satisfying the conditions in Theorem 2.1. Also, let assume the following assertions hold:*

a) the functionals $\int_{t_0}^{t_1} f_{ls}(t,x(t),\dot{x}(t))dt$, $l = \overline{1,p}$, $s = \overline{1,q}$, are (ρ_{ls}^1, b)-quasiinvex at $x^0(\cdot)$ with respect to η, θ;

b) $\int_{t_0}^{t_1} p(t)g(t,x(t),\dot{x}(t))dt$ is (ρ^2, b)-quasiinvex at $x^0(\cdot)$ with respect to η, θ;

c) $\int_{t_0}^{t_1} q(t)h(t,x(t),\dot{x}(t))dt$ is (ρ^3, b)-quasiinvex at $x^0(\cdot)$ with respect to η, θ;

d) one of the functionals $\int_{t_0}^{t_1} f_{ls}(t,x(t),\dot{x}(t))dt$, $l = \overline{1,p}$, $s = \overline{1,q}$, $\int_{t_0}^{t_1} p(t)g(t,x(t),\dot{x}(t))dt$, $\int_{t_0}^{t_1} q(t)h(t,x(t),\dot{x}(t))dt$ is strictly (ρ, b)-quasiinvex at $x^0(\cdot)$ with respect to η, θ ; ($\rho = \rho_{ls}^1$, ρ^2 or ρ^3 , respectively)

e) $\sum_{l=\overline{1,p};\ s=\overline{1,q}}\lambda_{ls}\rho_{ls}^1 + \rho^2 + \rho^3 \geq 0$ $(\rho_{ls}^1,\ \rho^2,\ \rho^3 \in R)$.

Then $x^0(\cdot)$ is an efficient solution for (MVP).

Proof. Assume that $x^0(\cdot)$ is not an efficient solution for (MVP). Then, for $l=\overline{1,p},\ s=\overline{1,q}$, there exists $x(\cdot)\in \mathrm{F}(I)$, such that

$$\int_{t_0}^{t_1} f_{ls}\,(t,x(t),\dot{x}(t))dt \le \int_{t_0}^{t_1} f_{ls}\,(t,x^0(t),\dot{x}^0(t))dt \quad (53)$$

and there exist at least $j\in\{1,2,\dots,p\},\ k\in\{1,2,\dots,q\}$ with

$$\int_{t_0}^{t_1} f_{jk}\,(t,x(t),\dot{x}(t))dt < \int_{t_0}^{t_1} f_{jk}\,(t,x^0(t),\dot{x}^0(t))dt. \quad (54)$$

Using a) (multiplying by $\lambda_{ls}\ge 0$ and making summation over $l=\overline{1,p}$ and $s=\overline{1,q}$), b) and c), we get (see $\lambda\frac{\partial f}{\partial x}(t,x^0,\dot{x}^0)$ as $\lambda_{ls}\frac{\partial f_{ls}}{\partial x}(t,x^0,\dot{x}^0)$ and $\lambda\frac{\partial f}{\partial \dot{x}}(t,x^0,\dot{x}^0)$ as $\lambda_{ls}\frac{\partial f_{ls}}{\partial \dot{x}}(t,x^0,\dot{x}^0)$, with summation over the repeated indices!)

$$b(x,x^0)\int_{t_0}^{t_1}\left[\eta(t,x,\dot{x}^0)\lambda\frac{\partial f}{\partial x}(t,x^0,\dot{x}^0)+\frac{d\eta}{dt}(t,x,\dot{x}^0)\lambda\frac{\partial f}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right]dt \quad (55)$$

$$\le -\left(\sum_{l=\overline{1,p};\ s=\overline{1,q}}\lambda_{ls}\,\rho_{ls}^1\right)b(x,x^0)\ \|\ \theta(x,x^0)\ \|^2,$$

$$b(x,x^0)\int_{t_0}^{t_1}\left[\eta(t,x,\dot{x}^0)p(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0)+\frac{d\eta}{dt}(t,x,\dot{x}^0)p(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right]dt \quad (56)$$

$$\le\ -\rho^2 b(x,x^0)\ \|\ \theta(x,x^0)\ \|^2,$$

$$b(x,x^0)\int_{t_0}^{t_1}\left[\eta(t,x,\dot{x}^0)q(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0)+\frac{d\eta}{dt}(t,x,\dot{x}^0)q(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right]dt \quad (57)$$

$$\le\ -\rho^3 b(x,x^0)\ \|\ \theta(x,x^0)\ \|^2.$$

Making the sum $(55)+(56)+(57)$, side by side, and taking into account d), we have

$$b(x,x^0)\int_{t_0}^{t_1}\eta\,(t,x,\dot{x}^0)\left[\lambda\frac{\partial f}{\partial x}(t,x^0,\dot{x}^0)+p(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0)+q(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0)\right]dt \quad (58)$$

$$+b(x,x^0)\int_{t_0}^{t_1}\frac{d\eta}{dt}(t,x,\dot{x}^0)\left[\lambda\frac{\partial f}{\partial \dot{x}}(t,x^0,\dot{x}^0)+p(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0)+q(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right]dt$$

$$<-\left(\sum_{l=\overline{1,p};\ s=\overline{1,q}}\lambda_{ls}\,\rho_{ls}^1+\rho^2+\rho^3\right)b(x,x^0)\ \|\ \theta(x,x^0)\ \|^2.$$

It follows that $b(x,x^0)>0$ and the foregoing inequality can be rewritten as

$$\int_{t_0}^{t_1}\eta\,(t,x,\dot{x}^0)\left[\lambda\frac{\partial f}{\partial x}(t,x^0,\dot{x}^0)+p(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0)+q(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0)\right]dt \quad (59)$$

$$+\int_{t_0}^{t_1}\frac{d\eta}{dt}(t,x,\dot{x}^0)\left[\lambda\frac{\partial f}{\partial \dot{x}}(t,x^0,\dot{x}^0)+p(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0)+q(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right]dt$$

$$<-\left(\sum_{l=\overline{1,p};\ s=\overline{1,q}}\lambda_{ls}\,\rho_{ls}^1+\rho^2+\rho^3\right)\|\ \theta(x,x^0)\ \|^2,$$

or, using the integration formula by parts, we get

$$\int_{t_0}^{t_1}\eta\,(t,x,\dot{x}^0)\left[\lambda\frac{\partial f}{\partial x}(t,x^0,\dot{x}^0)+p(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0)+q(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0)\right]dt \quad (60)$$

$$+\eta(t,x,\dot{x}^0)\left[\lambda\frac{\partial f}{\partial \dot{x}}(t,x^0,\dot{x}^0)+p(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0)+q(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right]\Big|_{t_0}^{t_1}$$

$$-\int_{t_0}^{t_1}\eta\,(t,x,\dot{x}^0)\frac{d}{dt}\left[\lambda\frac{\partial f}{\partial \dot{x}}(t,x^0,\dot{x}^0)+p(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0)+q(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right]dt$$

$$<-\left(\sum_{l=\overline{1,p};\ s=\overline{1,q}}\lambda_{ls}\,\rho_{ls}^1+\rho^2+\rho^3\right)\|\ \theta(x,x^0)\ \|^2.$$

Taking into account the boundary conditions $x(t_0)=x_0=x^0(t_0),\ x(t_1)=x_1=x^0(t_1)$ and knowing that $\eta(t,x^0(t),\dot{x}^0(t))=0,\ t\in I$ (see Definition 1.3), we obtain

$$\int_{t_0}^{t_1}\eta\,(t,x,\dot{x}^0)\left[\lambda\frac{\partial f}{\partial x}(t,x^0,\dot{x}^0)+p(t)\frac{\partial g}{\partial x}(t,x^0,\dot{x}^0)+q(t)\frac{\partial h}{\partial x}(t,x^0,\dot{x}^0)\right]dt \quad (61)$$

$$-\int_{t_0}^{t_1}\eta\,(t,x,\dot{x}^0)\frac{d}{dt}\left[\lambda\frac{\partial f}{\partial \dot{x}}(t,x^0,\dot{x}^0)+p(t)\frac{\partial g}{\partial \dot{x}}(t,x^0,\dot{x}^0)+q(t)\frac{\partial h}{\partial \dot{x}}(t,x^0,\dot{x}^0)\right]dt$$

$$<-\left(\sum_{l=\overline{1,p};\ s=\overline{1,q}}\lambda_{ls}\,\rho_{ls}^1+\rho^2+\rho^3\right)\|\ \theta(x,x^0)\ \|^2.$$

Using Theorem 2.1, we have

$$0<-\left(\sum_{l=\overline{1,p};\ s=\overline{1,q}}\lambda_{ls}\,\rho_{ls}^1+\rho^2+\rho^3\right)\|\ \theta(x,x^0)\ \|^2. \quad (62)$$

The hypothesis e) and $\|\ \theta(x,x^0)\ \|\ge 0$ lead to a contradiction. Therefore, x^0 is an efficient solution in (MVP) and the proof is complete.

If in Theorem 2.2 the functionals from the hypotheses b) and c) are replaced by the functional $\int_{t_0}^{t_1}[p(t)g(t,x(t),\dot{x}(t))+q(t)h(t,x(t),\dot{x}(t))]\,dt$, then the following result hold:

Corollary 2.1 (Sufficient efficiency conditions for (MVP)) *Let* $x^0(\cdot) \in \mathrm{F}(I)$, $\lambda \in R^{pq}$, $p: I \to R^m$, $q: I \to R^r$ *satisfying the conditions in Theorem* 2.1. *Also, let assume the following assertions hold:*

a) the functionals $\int_{t_0}^{t_1} f_{ls}(t, x(t), \dot{x}(t))dt$, $l = \overline{1,p}$, $s = \overline{1,q}$ *are* (ρ_{ls}^1, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η, θ;

b) $\int_{t_0}^{t_1} [p(t)g(t, x(t), \dot{x}(t)) + q(t)h(t, x(t), \dot{x}(t))]\, dt$ *is* (ρ^2, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η, θ;

c) one of the functionals $\int_{t_0}^{t_1} f_{ls}(t, x(t), \dot{x}(t))dt$, $l = \overline{1,p}$, $s = \overline{1,q}$, $\int_{t_0}^{t_1} [p(t)g(t, x(t), \dot{x}(t)) + q(t)h(t, x(t), \dot{x}(t))]\, dt$, *is strictly* (ρ, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η, θ; *(*$\rho = \rho_{ls}^1$ *or* ρ^2*, respectively)*

d) $\sum_{l=\overline{1,p};\, s=\overline{1,q}} \lambda_{ls}\, \rho_{ls}^1 + \rho^2 \geq 0 \quad (\rho_{ls}^1,\ \rho^2 \in R)$.

Then $x^0(\cdot)$ *is an efficient solution for* (MVP).

3. Conclusions

In the present paper, we have introduced and studied a new class of multiobjective variational problems subject to ordinary differential equations (ODEs) and/or ordinary differential inequations (ODIs) constraints. Within this framework, we have formulated and proved necessary and sufficient efficiency conditions for (*MVP*), developing a matrix optimization theory with many applications in engineering problems and mechanics. For other different but connected ideas to this subject, the reader is directed to [9], [10].

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