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Solution of Seventh Order Boundary Value Problem by Using Variational Iteration Method

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Abstract

In this paper, we introduce some basic idea of Variational iteration method for short (VIM) to solve the seventh order boundary value problems. It is to be mentioned that, presently, the literature on the numerical solutions of seventh order boundary value problem and associated eigen value problems is not available. By using a suitable transformation, the variational iteration method can be used to show that seventh order boundary value problems are equivalent to a system of integral equation. The VIM is used to solve effectively, easily, and accurately a large class of non-linear problems with approximations which converge rapidly to accurate solutions. For linear problems, it's exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. It is to be noted that the Lagrange multiplier reduces the iteration on integral operator and also minimizes the computational time. The method requires no transformation or linearization of any forms. Two numerical examples are presented to show the effectiveness and efficiency of the method. Also, we compare the result with exact solution. Finally, we investigate the error between numerical solution and exact solution and draw the graph of error function by using Mathematica.

Keywords

Variational Iteration Method, Boundary Value Problem, Mathematica, Exact Solution, Approximate Solution, Numerical Solution, Lagrange Multiplier, Absolute Error

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1. Introduction

In this paper, we consider the general seventh order boundary value problems of the type

$$\left.\begin{aligned} u^{(7)} &= f\big(x, u(x)\big), a \le x \le b \\ u^{(i)}(a) &= A_i, i = 0,1,2,3 \\ u^{(j)}(b) &= B_j, j = 0,1,2 \end{aligned}\right\} \quad (1)$$

Where $A_i, i = 0,1,2,3$ and $B_j, j = 0,1,2$ are finite real constants, also $f\big(x, u(x)\big)$ is a continuous function on $[a, b]$.

The theory of seventh order boundary value problems is not much available in the numerical analysis literature. These problems are generally arise in modelling induction motors with two rotor circuits. The induction motor behavior is represented by a fifth order differential equation model. This model contains two stator state variables, two rotor state variables and one shaft speed. Normally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters. To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth order and rotor impedance is algebraically altered as function of rotor speed under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when rotor frequency is not a single value. So, the behavior of such model show up in the seventh order [1].

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2. History of Variation Method (VIM)

The variational iteration method (VIM) developed in 1999 by He in [2–11] will be used to study the linear wave equation, nonlinear wave equation, and wave-like equation in bounded and unbounded domains. The method has been proved by many authors [12–23] to be reliable and efficient for a wide variety of scientific applications, linear and nonlinear as well. It was shown by many authors that this method is more powerful than existing techniques such as the Adomian method [24, 25], perturbation method, etc. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. The method is effectively used in [2–17] and the references therein. The perturbation method suffers from the computational workload, especially when the degree of nonlinearity increases. Moreover, the Adomian method suffers from the complicated algorithms used to calculate the Adomian polynomials that are necessary for nonlinear problems. The VIM has no specific requirements, such as linearization, small parameters, etc. for nonlinear operators.

The variational iteration method, which is a modified general Lagrange multiplier method [26] has been shown to solve effectively, easily and accurately, a large class of nonlinear problems with approximations which converge quickly to accurate solutions. It was successfully applied to autonomous ordinary differential equation [5], to nonlinear partial differential equations with variable coefficients [28], to Schrödinger-KDV, generalized KDV and shallow water equations [29], to Burgers' and coupled Burgers' equations [30], to the linear Helmoltz partial differential equation [31] and recently to nonlinear fractional differential equations with Caputo differential derivative [32], and other fields [2, 12, 35, 36-39]. On the other hand, one of the interesting topics among researchers is solving integro-differential equations.

3. Basic Ideas of Variational Iteration Method

To illustrate the basic concept and idea of the variational iteration technique, we consider the following general differential equation

$$Lf + Nf = g(x) \tag{2}$$

where L is a linear operator, N a nonlinear operator and g(x) is inhomogeneous forcing term. According to the variational iteration method [8–13], we can construct a correct functional as follows

$$f_{n+1}(x) = f_n(x) + \int_0^x \lambda\left(Lf_n(s) + L\widetilde{f_n}(s) - g(s)\right) ds \tag{3}$$

Where λ is a Lagrange multiplier, which can be identified optimally via the variational iteration method. The subscripts n denoted the nth approximation, ˜ un is considered as a restricted variation. i.e. $\delta\widetilde{f_n} = 0$; (3) is called as a correct functional.

The solution of linear problems can be solved in a single iteration step due to the exact identification of their Lagrange multiplier.

For the sake of simplicity, we consider the following system of differential equations:

$$x_i'(t) = u_i(t, x_i), i = 1,2,3, \dots\dots\dots.., n \tag{4}$$

subject to the boundary conditions, $x_i(0) = c_i$, i = 1, 2, 3,..., n.

To solve the system by means of the variational iteration method, we can rewrite the system (4) in the following form:

$$x_i'(t) = u_i(x_i) + g_i(t), i = 1,2,3, \dots\dots\dots.., n \tag{5}$$

subject to the boundary conditions, $x_i(0) = c_i$, i = 1, 2, 3,.., n and g_i is defined in (2).

The correct functional for the nonlinear system (1.04) can be expressed as follows:

$$\left.\begin{aligned} x_1^{(k+1)}(t) &= x_1^{(k)}(t) + \int_0^t \lambda_1\left(x_1'^{(k)}(T), f_1\left(\widetilde{x_1}^{(k)}(T), \widetilde{x_2}^{(k)}(T), \dots\dots.., \widetilde{x_n}^{(k)}(T)\right) - g_1(T)\right) dT \\ x_2^{(k+1)}(t) &= x_2^{(k)}(t) + \int_0^t \lambda_2\left(x_2'^{(k)}(T), f_2\left(\widetilde{x_1}^{(k)}(T), \widetilde{x_2}^{(k)}(T), \dots\dots.., \widetilde{x_n}^{(k)}(T)\right) - g_2(T)\right) dT \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. \\ x_n^{(k+1)}(t) &= x_n^{(k)}(t) + \int_0^t \lambda_n\left(x_n'^{(k)}(T), f_n\left(\widetilde{x_1}^{(k)}(T), \widetilde{x_2}^{(k)}(T), \dots\dots.., \widetilde{x_n}^{(k)}(T)\right) - g_n(T)\right) dT \end{aligned}\right\} \tag{6}$$

where $\lambda_i, i = 1,2,3, \dots\dots\dots, n$ are Lagrange multipliers, $\widetilde{x_1}, \widetilde{x_2}, \dots\dots\dots\dots, \widetilde{x_n}$ denote the restricted variations. Making the above functional stationary, we obtain the following conditions:

$$\lambda_i'(T)|_{T=t} = 0,$$

$$1 + \lambda_i'(T)|_{T=t} = 0$$

For $i = 1,2,3, \dots\dots\dots n$. Therefore, the Lagrange multipliers can be easily identified as:

$$\lambda_i = -1, i = 1,2,3, \dots\dots\dots, n \tag{7}$$

Substituting (7) into the correct functional (6), we have the following iteration formulae:

$$x_1^{(k+1)}(t) = x_1^{(k)}(t) - \int_0^t \left(x_1'^{(k)}(T), f_1\left(\widetilde{x_1}^{(k)}(T), \widetilde{x_2}^{(k)}(T), \dots \dots .., \widetilde{x_n}^{(k)}(T)\right) - g_1(T)\right) dT$$
$$x_2^{(k+1)}(t) = x_2^{(k)}(t) - \int_0^t \left(x_2'^{(k)}(T), f_2\left(\widetilde{x_1}^{(k)}(T), \widetilde{x_2}^{(k)}(T), \dots \dots .., \widetilde{x_n}^{(k)}(T)\right) - g_2(T)\right) dT$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots ..$$
$$x_n^{(k+1)}(t) = x_n^{(k)}(t) - \int_0^t \left(x_n'^{(k)}(T), f_n\left(\widetilde{x_1}^{(k)}(T), \widetilde{x_2}^{(k)}(T), \dots \dots .., \widetilde{x_n}^{(k)}(T)\right) - g_n(T)\right) dT$$

If we start the initial approximations $x_i(0) = c_i, i = 1,2,3, \dots .., n$, then the approximation can be completely determined; finally, we approximate the solution.

$$x_i(t) = \lim_{k \to \infty} x_i^{(k)}(t)$$

by the nth term $x_i^{(n)}(t)$ for $i = 1,2,3, \dots \dots \dots \dots \dots, n$.

4. Numerical Examples

In this paper, we present two examples to show efficiency and high accuracy of the variational iteration method for solving seventh order boundary value problems.

Example 01 Consider the following 7th order nonlinear boundary value problem

$$\left.\begin{aligned} f^{(7)}(x) = e^{-x} f^2(x), 0 < x < 1 \\ f(0) = f^{(1)}(0) = f^{(2)}(0) = f^{(3)}(0) = 1 \\ f(1) = f^{(1)}(1) = f^{(2)}(1) = e \end{aligned}\right\} \quad (8)$$

The exact solution of this problem is $f(x) = e^x$

The given 7th order boundary value problem can be transformed with the following system

$$\left.\begin{aligned} \frac{df}{dx} = l(x), \frac{dl}{dx} = m(x), \frac{dm}{dx} = n(x), \frac{dn}{dx} = p(x) \\ \frac{dp}{dx} = q(x), \frac{dq}{dx} = r(x), \frac{dr}{dx} = e^{-x} f^2(x) \end{aligned}\right\} \quad (9)$$

With

$$f(0) = 1, l(0) = 1, m(0) = 1, n(0) = 1, p(0) = A, q(0) = B, r(0) = C$$

The system of differential equation (9) can be written in terms of the following system of integral equations with Lagrange multipliers $\lambda_i = +1; i = 1,2,3, \dots \dots .,7$.

$$f^{(k+1)}(x) = 1 + \int_0^x l^{(k)}(t)dt$$

$$l^{(k+1)}(x) = 1 + \int_0^x m^{(k)}(t)dt$$

$$m^{(k+1)}(x) = 1 + \int_0^x n^{(k)}(t)dt$$

$$n^{(k+1)}(x) = 1 + \int_0^x p^{(k)}(t)dt$$

$$p^{(k+1)}(x) = A + \int_0^x q^{(k)}(t)dt$$

$$q^{(k+1)}(x) = B + \int_0^x r^{(k)}(t)dt$$

$$r^{(k+1)}(x) = C + \int_0^x e^{-t} f^2(t)dt$$

With

$$f(0) = 1, l(0) = 1, m(0) = 1, n(0) = 1, p(0) = A, q(0) = B, r(0) = C$$

Now,

$$f^{(1)}(x) = 1 + x$$

$$l^{(1)}(x) = 1 + x$$

$$m^{(1)}(x) = 1 + x$$

$$n^{(1)}(x) = 1 + Ax$$

$$p^{(1)}(x) = A + Bx$$

$$q^{(1)}(x) = B + Cx$$

$$f^{(2)}(x) = 1 + x + \frac{x^2}{2}$$

$$l^{(2)}(x) = 1 + x + \frac{x^2}{2}$$

$$m^{(2)}(x) = 1 + x + \frac{A}{2}x^2$$

$$n^{(2)}(x) = 1 + Ax + \frac{B}{2}x^2$$

$$p^{(2)}(x) = A + Bx + \frac{C}{2}x^2$$

$$f^{(3)}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$l^{(3)}(x) = 1 + x + \frac{x^2}{2} + \frac{A}{6}x^3$$

$$m^{(3)}(x) = 1 + x + \frac{A}{2}x^2 + \frac{B}{6}x^3$$

$$n^{(3)}(x) = 1 + Ax + \frac{B}{2}x^2 + \frac{C}{6}x^3$$

$$f^{(4)}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{A}{24}x^4$$

$$l^{(4)}(x) = 1 + x + \frac{x^2}{2} + \frac{A}{6}x^3 + \frac{B}{24}x^4$$

$$m^{(4)}(x) = 1 + x + \frac{A}{2}x^2 + \frac{B}{6}x^3 + \frac{C}{24}x^4$$

$$f^{(5)}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{A}{24}x^4 + \frac{B}{120}x^5$$

$$l^{(5)}(x) = 1 + x + \frac{x^2}{2} + \frac{A}{6}x^3 + \frac{B}{24}x^4 + \frac{C}{120}x^5$$

$$f^{(6)}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{A}{24}x^4 + \frac{B}{120}x^5 + \frac{C}{720}x^6$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

Hence, the series solution is given as

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{A}{24}x^4 + \frac{B}{120}x^5 + \frac{C}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9 + \frac{1}{3628800}x^{10} + \frac{2A-1}{39916800}x^{11} - \frac{1}{479001600}x^{12} - \frac{1}{6227020800}x^{13} + O(x^{14})$$

Using the boundary conditions at $x = 1$, we get the value of A, B and C as

$$A = 1.0000023410747971, B = 0.9999732900859826, C = 1.0000932820081587$$

Finally, the series solution can be written as

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1.0000023410747971}{24}x^4 + \frac{0.9999732900859826}{120}x^5 + \frac{1.0000932820081587}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9 + \frac{1}{3628800}x^{10} + \frac{1.00001865640163}{39916800}x^{11} - \frac{1}{479001600}x^{12} - \frac{1}{6227020800}x^{13} + O(x^{14})$$

Table 1. Comparison of numerical results for example 01.

x	Exact Solution	Approximate Solution	Absolute Error
0	1	1	0
0.1	1.105171	1.105171	7.65E-12
0.2	1.221403	1.221403	9.31E-11
0.3	1.349859	1.349859	3.43E-10
0.4	1.491825	1.491825	7.48E-10
0.5	1.648721	1.648721	1.16E-9
0.6	1.822119	1.822119	1.36E-9
0.7	2.013753	2.013753	1.19E-9
0.8	2.225541	2.225541	6.76E-10
0.9	2.459603	2.459603	1.55E-10
1.0	2.718281	2.718282	1.21E-11
1.1	3.004166	3.004166	3.94E-10
1.2	3.320110	3.320117	5.54E-9

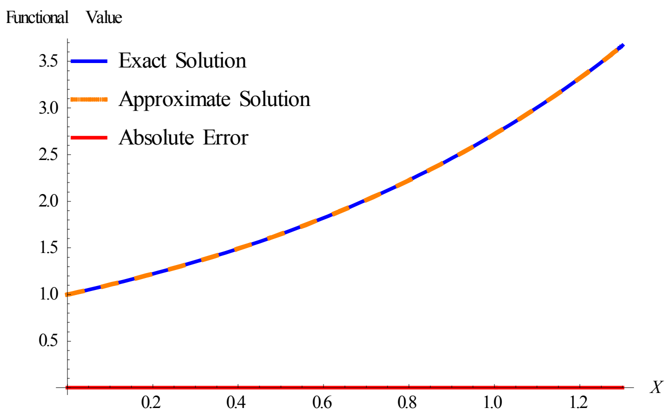

Figure 1. Comparison of the approximate solution with exact solution for example 01.

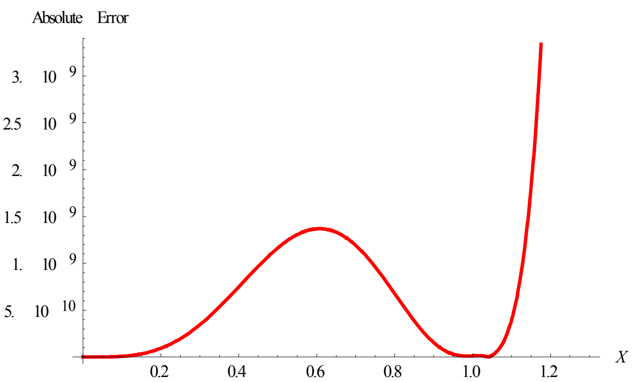

Figure 2. Absolute error for example 01.

Example 02 Consider the following 7th order nonlinear boundary value problem

$$\left.\begin{array}{c} f^{(7)}(x) = -7e^x + u(x), 0 \le x \le 1 \\ f(0) = 1, f^{(1)}(0) = 0, f^{(2)}(0) = -1, f^{(3)}(0) = -2 \\ f(1) = 0, f^{(1)}(1) = -e, f^{(2)}(1) = -2e \end{array}\right\} \quad (10)$$

The exact solution of this problem is $f(x) = (1 - x)e^x$

The given 7th order boundary value problem can be transformed with the following system

$$\left.\begin{array}{c} \frac{df}{dx} = l(x), \frac{dl}{dx} = m(x), \frac{dm}{dx} = n(x), \frac{dn}{dx} = p(x) \\ \frac{dp}{dx} = q(x), \frac{dq}{dx} = r(x), \frac{dr}{dx} = f(x) - 7e^x \end{array}\right\} \quad (11)$$

With

$$f(0) = 1, l(0) = 0, m(0) = -1, n(0) = -2, p(0) = A, q(0) = B, r(0) = C$$

The system of differential equation (11) can be written in terms of the following system of integral equations with Lagrange multipliers $\lambda_i = +1; i = 1,2,3, \dots\dots, 7$.

$$f^{(k+1)}(x) = 1 + \int_0^x l^{(k)}(t)dt$$

$$l^{(k+1)}(x) = 0 + \int_0^x m^{(k)}(t)dt$$

$$m^{(k+1)}(x) = -1 + \int_0^x n^{(k)}(t)dt$$

$$n^{(k+1)}(x) = -2 + \int_0^x p^{(k)}(t)dt$$

$$p^{(k+1)}(x) = A + \int_0^x q^{(k)}(t)dt$$

$$q^{(k+1)}(x) = B + \int_0^x r^{(k)}(t)dt$$

$$r^{(k+1)}(x) = C + \int_0^x (u(t) - 7e^t)dt$$

With

$$f(0) = 1, l(0) = 0, m(0) = -1, n(0) = -2, p(0) = A, q(0) = B, r(0) = C$$

Now,

$$f^{(1)}(x) = 1$$

$$l^{(1)}(x) = -x$$

$$m^{(1)}(x) = -1 - 2x$$

$$n^{(1)}(x) = -2 + Ax$$

$$p^{(1)}(x) = A + Bx$$

$$q^{(1)}(x) = B + Cx$$

$$f^{(2)}(x) = 1 - \frac{x^2}{2}$$

$$l^{(2)}(x) = -x - x^2$$

$$m^{(2)}(x) = -1 - 2x + \frac{A}{2}x^2$$

$$n^{(2)}(x) = -2 + Ax + \frac{B}{2}x^2$$

$$p^{(2)}(x) = A + Bx + \frac{C}{2}x^2$$

$$f^{(3)}(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3}$$

$$l^{(3)}(x) = -x - x^2 + \frac{A}{6}x^3$$

$$m^{(3)}(x) = -1 - 2x + \frac{A}{2}x^2 + \frac{B}{6}x^3$$

$$n^{(3)}(x) = -2 + Ax + \frac{B}{2}x^2 + \frac{C}{6}x^3$$

$$f^{(4)}(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} + \frac{A}{24}x^4$$

$$l^{(4)}(x) = -x - x^2 + \frac{A}{6}x^3 + \frac{B}{24}x^4$$

$$m^{(4)}(x) = -1 - 2x + \frac{A}{2}x^2 + \frac{B}{6}x^3 + \frac{C}{24}x^4$$

$$f^{(5)}(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} + \frac{A}{24}x^4 + \frac{B}{120}x^5$$

$$l^{(5)}(x) = -x - x^2 + \frac{A}{6}x^3 + \frac{B}{24}x^4 + \frac{C}{120}x^5$$

$$f^{(6)}(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} + \frac{A}{24}x^4 + \frac{B}{120}x^5 + \frac{C}{720}x^6$$

⋮ ⋮ ⋮ ⋮ ⋮⋮ ⋮ ⋮⋮ ⋮ ⋮⋮ ⋮ ⋮ ⋮ ⋮ ⋮⋮ ⋮ ⋮ ⋮

Hence, the series solution is given as

$$f(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} + \frac{A}{24}x^4 + \frac{B}{120}x^5 + \frac{C}{720}x^6 - \frac{1}{840}x^7 - \frac{1}{5760}x^8 - \frac{1}{45360}x^9 - \frac{1}{403200}x^{10} + \frac{A-7}{39916800}x^{11} - \frac{B-7}{479001600}x^{12} + \frac{C-7}{6227020800}x^{13} + O(x^{14})$$

Using the boundary conditions at $x = 1$, we get the value of A, B and C as

$$A = -3.0000000000125264, B = -3.999999999873768, C = -5.000000000395227$$

Finally, the series solution can be written as

$$f(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{3.0000000000125264}{24}x^4 - \frac{3.999999999873768}{120}x^5 - \frac{5.000000000395227}{720}x^6 - \frac{1}{840}x^7 - \frac{1}{5760}x^8 - \frac{1}{45360}x^9 - \frac{1}{403200}x^{10} - \frac{10.0000000000125264}{39916800}x^{11} + \frac{10.999999999873768}{479001600}x^{12} - \frac{12.000000000395227}{6227020800}x^{13} + O(x^{14})$$

Table 2. Comparison of numerical results for example 02.

x	Exact Solution	Approximate Solution	Absolute Error
0	1	1	0
0.1	0.994653	0.994653	1.11E-16
0.2	0.977122	0.977122	4.44E-16
0.3	0.944901	0.944901	2.23E-14
0.4	0.895094	0.895094	7.66E-13
0.5	0.824361	0.824361	1.12E-11
0.6	0.728847	0.728847	1.00E-10
0.7	0.604125	0.604125	6.36E-10
0.8	0.445108	0.445108	3.16E-9
0.9	0.245960	0.245960	1.30E-8
1.0	0	4.61E-8	4.60E-8

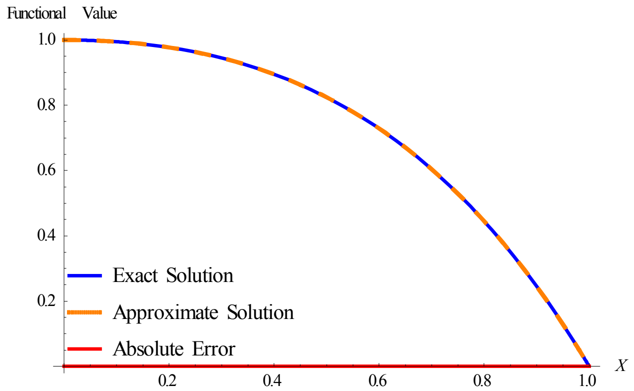

Figure 3. Comparison of the approximate solution with exact solution for example 02.

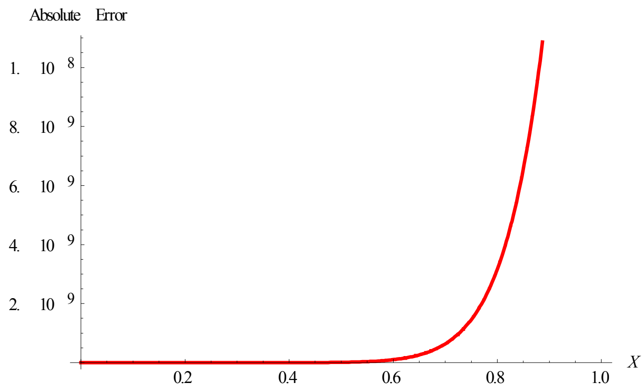

Figure 4. Absolute error for example 02.

5. Conclusion

In this paper we have considered seventh order boundary value problem. First, we have shown that fundamentals of variational iteration method. After that we solved two numerical examples by using variational iteration method. then, we show that the approximate solution and compare it with exact solution. We see that, the approximate solution is a series solution and it converse to the exact solution. Finally, we draw a graph of comparison between approximate solution and exact solution and also a graph for absolute error by using Mathematica. Finally, we conclude that the numerical examples show the high degree of efficiency of the variational iteration method.

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