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Approximate Solutions of Fifth Order More Critically Damped Nonlinear Systems with Triply Equal Eigenvalues

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Abstract

In this paper, an extension of the Krylov-Bogoliubov-Mitropolskii (KBM) method (which is also regarded as one of the most convenient and widely used methods for investigating the transient behavior of nonlinear systems) is used to figure out the solutions of fifth order more critically damped nonlinear systems. To this end, the analytical approximate solutions of fifth more critically damped nonlinear systems are considered in which the three eigenvalues are identical and another two are different. In this article, we suggest that the perturbation solutions obtained by the extended KBM method adequately matches up with the numerical solutions.

Keywords

KBM Method, Analytical Solution, More Critically Damped, Nonlinearity, Eigenvalues

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1. Introduction

Physicists, engineers and applied mathematicians have long been confronted with difficulties such as nonlinear governing equations, variable coefficients, and nonlinear boundary conditions at complex known or unknown boundaries, which prevent them solving these difficulties accurately. As a result, they are compelled to recourse to numerical solutions, or approximation solutions, or a combination of both. Of these approximation solutions, the systematic method of perturbations is firmly established that actually formed the basis of the widely recognized method named the Krylov-Bogoliubov-Mitropolskii (KBM) [1, 2] method. Generally, the KBM method is applied to study nonlinear oscillatory and non-oscillatory differential systems with small nonlinearities. It was first expounded by Krylov and Bogoliubov [1] to find periodic solutions of second order nonlinear differential systems with small nonlinearities. Consequently, this method was expanded by Popov [3] to damped oscillatory processes where a strong linear damping force was present. Later, it was amplified and justified by Bogoliubov and Mitropolskii [2]. However, Popov's results were rediscovered by Mendelson [4] who took into account the physical significance of the damped oscillatory systems. Murty and Deekshatulu [5] then extended it to over–damped nonlinear systems. Ultimately, a unified KBM method for second order nonlinear systems was anticipated by Murty [6] that covered all the undamped, over-damped and damped oscillatory cases. In the meantime, Osiniskii [7] established it for the first time to investigate third order nonlinear differential systems. However, Osiniskii [7] imposed some restrictions that rendered the solution over-simplified. Later, Mulholland [8] came forward to lift those restrictions and found the envisioned solutions for third order nonlinear systems. Thereafter, solutions were offered by Bojadziv [9] with respect to nonlinear systems through the conversion of

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the method to damped nonlinear oscillations for a 3-dimensional differential system. Afterwards, the time dependent third-order oscillating systems with damping were developed by Alam and Sattar [10]. It was then generalized by Akbar *et al.* [11] in order to make it less intricate than the method propounded by Murty *et al.* [12]. Next, it was extended further by Akbar *et al.* [13] to fourth order damped oscillatory systems in the case when the four eigenvalues were complex conjugate. Following that, a technique for fourth order more critically damped nonlinear systems was devised by Rahaman [14]. In due course, Kawser *et al.* [15] put forward analytical solutions of fourth order critically undamped oscillatory nonlinear systems with pairwise equal imaginary eigenvalues. Kawser *et al.* [16] also set forth a new analytical solution of fourth order critically damped nonlinear oscillatory systems in the case when the eigenvalues were pairwise equal and complex. Lately, Alam *et al.* [17] framed perturbation solutions of fourth order more critically damped nonlinear systems in the case of four equal eigenvalues. Further, Kawser *et al.* [18] spelt out asymptotic solutions of fifth order more critically damped nonlinear systems in the case of four repeated roots. Furthermore, Alam *et al.* [19] stepped forward with an asymptotic solution for the fifth order critically damped nonlinear systems in the case for small equal eigenvalues. In addition, analytical approximate solutions of fifth order more critically damped nonlinear systems were proposed by Rahaman and Kawser [20]. Besides, perturbation solutions of fifth order critically undamped nonlinear oscillatory systems with pairwise equal eigenvalues were suggested by Kawser *et al.* [21]. On top of that, the Krylov-Bogoliubov-Mitropolskii method for fifth order critically damped nonlinear systems in the case for large equal eigenvalues was expounded by Bagchi *et al.* [22].

In this paper, we study the solutions of fifth order more critically damped nonlinear systems when three of the eigenvalues are equal and another two distinct. Lastly, we propose that the obtained perturbation results satisfactorily match up with the numerical results for different sets of initial conditions as well as different sets of eigenvalues. In this study, *Mathematica 9.0* is used to compute all the calculation and results.

2. Method

Consider a fifth order nonlinear system governed by the ordinary differential equation

$$x^{(v)} + k_1 x^{(iv)} + k_2 \dddot{x} + k_3 \ddot{x} + k_4 \dot{x} + k_5 x = -\varepsilon f(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}) \quad (1)$$

where $x^{(v)}$ and $x^{(iv)}$ stands for the fifth and fourth derivative and over dots are used for the first, second and third derivatives of x with respect to t; k_1, k_2, k_3, k_4, k_5 are characteristic parameters, ε is a small parameter and $f\left(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}\right)$ is the given nonlinear function. As the system (1) is of fifth order more critically damped, we consider the equation (1) which has three equal eigenvalues and another two are distinct. We assume that the eigenvalues are $-\lambda, -\lambda, -\lambda, -\mu$ and $-\eta$.

Thus, when $\varepsilon = 0$, the solution of the corresponding linear equation of (1) is

$$x(t,0) = (a_0 + b_0 t + c_0 t^2) e^{-\lambda t} + d_0 e^{-\mu t} + h_0 e^{-\eta t} \quad (2)$$

where a_0, b_0, c_0, d_0, h_0 are constants of integration.

However, if $\varepsilon \neq 0$, following Alam [23], we choose an asymptotic solution of (1) in the form

$$x(t,\varepsilon) = (a + bt + ct^2) e^{-\lambda t} + d e^{-\mu t} + h e^{-\eta t} + \varepsilon u_1(a,b,c,d,h,t) \ldots \quad (3)$$

where a, b, c, d, h are functions of t and they satisfy the following first order differential equations

$$\begin{aligned} \dot{a}(t) &= \varepsilon A_1(a, b, c, d, h, t) + \cdots \\ \dot{b}(t) &= \varepsilon B_1(a, b, c, d, h, t) + \cdots \\ \dot{c}(t) &= \varepsilon C_1(a, b, c, d, h, t) + \cdots \\ \dot{d}(t) &= \varepsilon D_1(a, b, c, d, h, t) + \cdots \\ \dot{h}(t) &= \varepsilon H_1(a, b, c, d, h, t) + \cdots \end{aligned} \quad (4)$$

Here, the first few terms in the series expansion of (3) and (4) are considered. Thereafter, we compute the functions u_i and A_i, B_i, C_i, D_i, H_i for $i = 1,2,3,...$ such that a, b, c, d, h appearing in (3) and (4) satisfy the given differential equation (1). Please note that, in order to ascertain these unknown functions, it is usual in the KBM method that the correction terms, u_i for $i = 1,2,3,...$ must ignore the terms (known as secular terms) which make them large. Theoretically, the solution can be achieved up to the accuracy of any order of approximation. But, owing to the fast growing algebraic intricacy for the derivation of the formulae, the solution is generally limited to a lower order, usually the first as suggested by Murty [6].

Now differentiating the equation (3) five times with respect to t, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}, x^{(v)}$ in the original equation (1), using the relations presented in (4), and, finally, equating the coefficients of ε, we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+t\frac{\partial^2 B_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+t^2\frac{\partial^2 C_1}{\partial t^2}+6t\frac{\partial C_1}{\partial t}+6C_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3$$
$$\left(\frac{\partial}{\partial t}+\eta-\mu\right)D_1+e^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^3\left(\frac{\partial}{\partial t}+\mu-\eta\right)H_1+\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial}{\partial t}+\eta\right)u_1=-f^{(0)}(a,b,c,d,h,t) \quad (5)$$

where $f^{(0)}\left(a,\, b,\, c,\, d,\, h,\, t\right)=f\left(x_0,\, \dot{x}_0,\, \ddot{x}_0,\, \dddot{x}_0,\, x_0^{(iv)}\right)$ and $x_0=(a+bt+ct^2)e^{-\lambda t}+de^{-\mu t}+he^{-\eta t}$

We have expanded the function $f^{(0)}$ in the Taylor's series (see also Sattar [24], Alam [25], Alam and Sattar [26] for particulars) about the origin in power of t.

Therefore, we obtain

$$f^{(0)}=\sum_{q=0}^{\infty}t^q\sum_{i,j,k,m=0}^{\infty}F_{q,m}(a,b,c,d,h)\,e^{-(i\lambda+j\mu+k\eta)t} \quad (6)$$

Here, $F_{q,m}$ is function of a, b, c, d, h and the limit of $i,\ j,\ k, m$ are vary from 0 to ∞, however, for a particular problem they have some definite values. Thus, using equation (6), equation (5) becomes

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+t\frac{\partial^2 B_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+t^2\frac{\partial^2 C_1}{\partial t^2}+6t\frac{\partial C_1}{\partial t}+6C_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3$$
$$\left(\frac{\partial}{\partial t}+\eta-\mu\right)D_1+e^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^3\left(\frac{\partial}{\partial t}+\mu-\eta\right)H_1+\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial}{\partial t}+\eta\right)u_1 \quad (7)$$
$$=-\sum_{q=0}^{\infty}t^q\sum_{i,j,k,m=0}^{\infty}F_{q,m}(a,b,c,d,h)\,e^{-(i\lambda+j\mu+k\eta)t}$$

Following the KBM method, Murty *et al.* [12], Sattar [24], Alam and Sattar [25] impose the condition that u_1 does not contain the fundamental terms of $f^{(0)}$. Thus, equation (7) can be separated for unknown functions A_1, B_1, C_1, D_1, H_1 and u_1 in the following way:

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+t\frac{\partial^2 B_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+t^2\frac{\partial^2 C_1}{\partial t^2}+6t\frac{\partial C_1}{\partial t}+6C_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3$$
$$\left(\frac{\partial}{\partial t}+\eta-\mu\right)D_1+e^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^3\left(\frac{\partial}{\partial t}+\mu-\eta\right)H_1=-\sum_{q=0}^{2}t^q\sum_{i,j,k,m=0}^{\infty}F_{q,m}(a,b,c,d,h)\,e^{-(i\lambda+j\mu+k\eta)t} \quad (8)$$

$$\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial}{\partial t}+\eta\right)u_1=-\sum_{q=3}^{\infty}t^q\sum_{i,j,k,m=0}^{\infty}F_{q,m}(a,b,c,d,h)\,e^{-(i\lambda+j\mu+k\eta)t} \quad (9)$$

Now, comparing the coefficients of t^0, t^1 and t^2 from both sides of equation (8), we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2}+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3\left(\frac{\partial}{\partial t}+\eta-\mu\right)D_1+e^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^3$$
$$\left(\frac{\partial}{\partial t}+\mu-\eta\right)H_1=-\sum_{i,j,k,m=0}^{\infty}F_{0,m}(a,b,c,d,h)\,e^{-(i\lambda+j\mu+k\eta)t} \quad (10)$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2}+\frac{\partial C_1}{\partial t}\right)=-\sum_{i,j,k,m=0}^{\infty}F_{1,m}(a,b,c,d,h)\,e^{-(i\lambda+j\mu+k\eta)t} \quad (11)$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 C_1}{\partial t^2}=-\sum_{i,j,k,m=0}^{\infty}F_{2,m}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\eta)t} \tag{12}$$

Here, there are three equations (10) to (12) for ascertain the unknown functions A_1, B_1, C_1, D_1, H_1. Therefore, in order to find the unknown functions A_1, B_1, C_1, D_1, H_1 some well-known operator method must be applied. We can assess the value of C_1 from equation (12). From equation (11) we can separate the unknown functions C_1 and B_1; and solve for B_1. In equation (10), applying some well-known operator method to separate the unknown functions A_1, D_1 and H_1, then solve them for unknown functions A_1, D_1 and H_1.

Since $\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{h}$ are proportional to small parameter ε, they are gradually varying functions of time t and, for first approximate solution, we may regard them as constants in the right hand side. Murty and Deekshatulu [5], and Murty *et al.* [12] fist made this assumption. The solutions of the equation (4), therefore, become

$$\begin{aligned}
&a=a_0+\varepsilon\int_0^t A_1(a,\,b,\,c,\,d,\,h,\,t)\,dt; b=b_0+\varepsilon\int_0^t B_1(a,\,b,\,c,\,d,\,h,\,t)\,dt;\\
&c=c_0+\varepsilon\int_0^t C_1(a,\,b,\,c,\,d,\,h,\,t)\,dt; d=d_0+\varepsilon\int_0^t D_1(a,\,b,\,c,\,d,\,h,\,t)\,dt;\\
&h=h_0+\varepsilon\int_0^t H_1(a,\,b,\,c,\,d,\,h,\,t)\,dt
\end{aligned} \tag{13}$$

Please further note that equation (9) is an inhomogeneous linear ordinary differential equation. Thus, it can be solved by the well-known operator method. We obtain the complete solution of (1) substituting the values of a, b, c, d, h and u_1 in the equation (3), Thus, the determination of the first approximate solution is complete.

3. Example

As an example of the above procedure, we consider the Duffing type equation of fifth order nonlinear differential system

$$x^{(v)}+k_1x^{(iv)}+k_2\dddot{x}+k_3\ddot{x}+k_4\dot{x}+k_5x=-\varepsilon x^3 \tag{14}$$

Comparing equation (1) with equation (14), we obtain $f(x,\dot{x},\ddot{x},\dddot{x},x^{(iv)})=x^3$.

Therefore,

$$\begin{aligned}
f^{(0)}=&3\left(\frac{1}{3}a^3+a^2bt+ab^2t^2+a^2ct^2+b^3t^3+2abct^3+b^2ct^4+ac^2t^4+bc^2t^5+c^3\right)e^{-3\lambda t}+3\big(a^2d+2abdt\\
&+2acdt^2+b^2bt^2+2bcdt^3+c^2dt^4\big)e^{-(2\lambda+\mu)t}+3\left(a^2h+2abht+b^2ht^2+2acht^2+2bcht^3+c^2ht^4\right)e^{-(2\lambda+\eta)t}\\
&+3\left(ad^2+bd^2t+b^2ht^2+cd^2t^2\right)e^{-(\lambda+2\mu)t}+6\left(adh+bdht+cdht^2\right)e^{-(\lambda+\mu+\eta)t}+3dh^2e^{-(\mu+2\eta)t}+d^3e^{-3\mu t}\\
&+3\left(ah^2+bh^2t+ch^2t^2\right)e^{-(\lambda+2\eta)t}\quad+3d^2he^{-(2\mu+\eta)t}+h^3e^{-3\eta t}
\end{aligned} \tag{15}$$

For equation (14), the equations (9) to (12) respectively becomes

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial}{\partial t}+\eta\right)u_1=-\Big\{&\left(b^3+6abc+3b^2ct+3ac^2t+3bc^2t^2+c^3\,t^3\right)t^3e^{-3\lambda t}\\
&+3\left(2bcd+c^2dt\right)t^3e^{-(2\lambda+\mu)t}+3\left(2bch+c^2ht\right)t^3e^{-(2\lambda+\eta)t}\Big\}
\end{aligned} \tag{16}$$

$$\begin{aligned}
&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2}+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3\left(\frac{\partial}{\partial t}+\eta-\mu\right)D_1+e^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^3\left(\frac{\partial}{\partial t}+\mu-\eta\right)H_1\\
&=-a^3e^{-3\lambda t}-d^3e^{-3\mu t}-3ad^2e^{-(\lambda+2\mu)t}-3a^2de^{-(2\lambda+\mu)t}-3a^2he^{-(2\lambda+\eta)t}-3d^2he^{-(2\mu+\eta)t}\\
&\quad-6bdhe^{-(\lambda+\mu+\eta)t}-3ah^2e^{-(\lambda+2\eta)t}-3dh^2e^{-(\mu+2\eta)t}-h^3e^{-3\eta t}
\end{aligned} \tag{17}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2}+\frac{\partial C_1}{\partial t}\right)=-3\Big(a^2b\,e^{-3\lambda t}+bd^2e^{-(\lambda+2\mu)t}+2abde^{-(2\lambda+\mu)t}$$
$$+2abhe^{-(2\lambda+\eta)t}+bh^2e^{-(\lambda+2\eta)t}+2bdhe^{-(\lambda+\mu+\eta)t}\Big) \tag{18}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 C_1}{\partial t^2}=-\Big\{3\left(ab^2+a^2c\right)e^{-3\lambda t}+3\left(2acd+b^2d\right)e^{-(2\lambda+\mu)t}+3cd^2e^{-(\lambda+2\mu)t}$$
$$+3\left(2ach+b^2h\right)e^{-(2\lambda+\eta)t}+3ch^2e^{-(\lambda+2\eta)t}+6cdhe^{-(\lambda+\mu+\eta)t}\Big\} \tag{19}$$

Solving equations (19), we achieve

$$C_1=\left(l_1ab^2+l_2a^2c\right)e^{-2\lambda t}+\left(l_3acd+l_4b^2d\right)e^{-(\lambda+\mu)t}+\left(l_5ach+l_6b^2h\right)e^{-(\lambda+\eta)t}+l_7cd^2e^{-2\mu t}+l_8ch^2e^{-2\eta t}+l_9cdhe^{-(\mu+\eta)t} \tag{20}$$

where

$$l_1=\frac{-3}{4\lambda^2(3\lambda-\eta)(3\lambda-\mu)},\ l_2=l_1,\ l_3=\frac{-3}{\lambda(\lambda+\mu)^2(2\lambda+\mu-\eta)},\ l_4=\frac{l_3}{2},$$

$$l_5=\frac{-3}{\lambda(\lambda+\eta)^2(2\lambda+\eta-\mu)},\ l_6=\frac{l_4}{2},\ l_7=\frac{-3}{4\mu^2(\lambda+\mu)(2\mu+\lambda-\eta)},$$

$$l_8=\frac{-3}{4\eta^2(\lambda+\eta)(\lambda+2\eta-\mu)},\ l_9=\frac{-3}{(\lambda+\eta)(\mu+\eta)^2(\lambda+\mu)}$$

Substituting the value of C_1 from equation (20) in equation (18), we obtain

$$B_1=\left(m_1a^2b+m_2ab^2+m_3a^2c\right)e^{-2\lambda t}+\left(m_4abh+m_5b^2h+m_6ach\right)e^{-(\lambda+\eta)t}+\left(m_7bd^2+m_8cd^2\right)e^{-2\mu t}$$
$$+\left(m_9bdh+m_{10}cdh\right)e^{-(\mu+\eta)t}+\left(m_{11}abd+m_{12}b^2d+m_{13}acd\right)e^{-(\lambda+\mu)t}+\left(m_{14}bh^2+m_{15}ch^2\right)e^{-2\eta t} \tag{21}$$

where

$$m_1=\frac{3}{4\lambda^2(3\lambda-\mu)(3\lambda-\eta)},\ m_2=\frac{3(11\lambda\eta-39\lambda^2-3\mu\eta+11\lambda\mu)}{4\lambda^3(3\lambda-\mu)^2(3\lambda-\eta)^2},\ m_3=m_2,$$

$$m_4=\frac{3}{\lambda(\lambda+\eta)^2(2\lambda+\eta-\mu)},\ m_5=\frac{-3(\eta^2+11\lambda\eta+16\lambda^2-\mu\eta-7\lambda\mu)}{2\lambda^2(\lambda+\eta)^3(2\lambda+\eta-\mu)^2},$$

$$m_6=2m_5,\ m_7=\frac{3}{4\mu^2(\lambda+\mu)(\lambda+2\mu-\eta)},\ m_8=\frac{3(3\lambda\eta-3\lambda^2+5\mu\eta-13\lambda\mu-12\mu^2)}{4\mu^3(\lambda+\mu)^2(\lambda+2\mu-\eta)^2},$$

$$m_9=\frac{6}{(\mu+\eta)(\lambda+\mu)^2(\lambda+\eta)},\ m_{11}=\frac{6}{\lambda(\lambda+\mu)^2(2\lambda+\mu-\eta)},$$

$$m_{12}=-\frac{3(16\lambda^2+11\lambda\mu+\mu^2-7\mu\eta-7\lambda\eta)}{2\lambda^2(2\lambda+\mu-\eta)^2(\lambda+\mu)^3},\ m_{13}=2m_{12},$$

$$m_{14}=\frac{3}{4\eta^2(\lambda+\eta)(\lambda+2\eta-\mu)},\ m_{15}=\frac{3(3\lambda\mu-3\lambda^2-12\eta^2-13\lambda\eta+5\mu\eta)}{4\eta^2(\lambda+\eta)(\lambda+2\eta-\mu)}$$

Thus, we obtain imposing the condition $\lambda<<\mu<<\eta$ and separating the equation (2.2.4) for A_1, D_1 and H_1 using the operator method

$$A_1 = \left(p_1 a^2 b + p_2 ab^2 + p_3 a^3 + p_4 a^2 c \right) e^{-2\lambda t} \tag{22}$$

where

$$p_1 = \frac{3(39\lambda^2 - 11\lambda\eta + 3\mu\eta - 11\lambda\mu)}{8\lambda^3(3\lambda-\mu)^2(3\lambda-\eta)^2},\ p_2 = \frac{1}{4\lambda^2(\mu-3\lambda)(3\lambda-\eta)},\ p_3 = \frac{1}{2}p_2,$$

$$p_4 = \frac{3}{4\lambda^4(\mu-3\lambda)^3(3\lambda-\eta)^3}\{\lambda\eta(-261\lambda^2+146\lambda\mu-21\mu^2)+\eta^2(38\lambda^2-21\lambda\mu+3\mu^2) + \lambda^2(459\lambda^2-261\lambda\mu+38\mu^2)\}$$

$$D_1 = q_1 d^3 e^{-2\mu t} + \left(q_2 ad^2 + q_3 bd^2 + q_4 cd^2 \right) e^{-(\lambda+\mu)t} + \left(q_5 abd + q_6 b^2 d + q_7 a^2 d + q_8 acd \right) e^{-2\lambda t} \tag{23}$$

where

$$q_1 = \frac{1}{(3\mu-\eta)(\lambda-3\mu)^3},\ q_2 = \frac{6}{16\mu^4(\lambda+\mu)(\eta-2\mu-\lambda)},$$

$$q_3 = \frac{3}{16\mu^5(\lambda+\mu)^2(\eta-2\mu-\lambda)^3}\{12\mu^2 - 3\lambda\eta + 3\lambda^3 - 5\mu\eta + 13\lambda\mu\},$$

$$q_4 = \frac{3}{8\mu^5(\mu+\lambda)^2(\eta-2\mu-\lambda)^3}\{3\lambda^4 + 24\lambda^3\mu + 72\lambda^2\mu^2 + 93\lambda\mu^3 + 44\mu^4 + \eta^2(3\lambda^2+9\lambda\mu+8\mu^2) - \eta(6\lambda^3+33\lambda^2\mu+60\lambda\mu^2+37\mu^3)\},$$

$$q_5 = \frac{3}{\lambda^2(\mu+\lambda)^5(\eta-2\mu-\lambda)^3}\{-\lambda(\lambda+\mu)(32\lambda^3+38\lambda^2\mu+13\lambda\mu^2+\mu^3 + \eta^2(7\lambda+\mu) - 2\eta(15\lambda^2+10\lambda\mu+\mu^2)\},$$

$$q_6 = \frac{3}{2\lambda^2(\mu+\lambda)^5(\eta-2\mu-\lambda)^3}\{156\lambda^4+210\lambda^3\mu+91\lambda^2\mu^2+14\lambda\mu^3+\mu^4 + \eta^2(31\lambda^2+8\lambda\mu+\mu^2) - 2\eta(69\lambda^3+55\lambda^2\mu+11\lambda\mu^2+\mu^3)\},$$

$$q_7 = \frac{3}{(\mu+\lambda)^3(\eta-2\mu-\lambda)},\quad q_8 = q_6$$

And

$$\begin{aligned} H_1 &= s_1 h^3 e^{-2\eta t} + s_2 dh^2 e^{-(\mu+\eta)t} + s_3 d^2 h e^{-2\mu t} + \left(s_4 cdh + s_5 adh + s_6 bdh \right) e^{-(\lambda+\mu)t} \\ &+ \left(s_7 ch^2 + s_8 ah^2 + s_9 bh^2 \right) e^{-(\lambda+\eta)t} + \left(s_{10} abh + s_{11} b^2 h + s_{12} a^2 h + s_{13} ach \right) e^{-2\lambda t} \end{aligned} \tag{24}$$

where

$$s_1 = \frac{1}{(\mu-3\eta)(3\eta-\lambda)^3},\ s_2 = -\frac{3}{2\eta(2\eta+\mu-\lambda)^3},\ s_3 = -\frac{3}{(\mu+\eta)(\eta+2\mu-\lambda)^3},$$

$$s_4 = \frac{12}{(\lambda+\eta)^3(\mu+\lambda)^2(\eta+\mu)^5}\{\eta^4 + 6\lambda^4 + 18\lambda^3\mu + 18\lambda^2\mu^2 + 6\lambda\mu^3 + \mu^4 + 6\eta^3(\lambda+\mu) + 2\eta^2(9\lambda^2+18\lambda\mu+8\mu^2) + 6\eta(3\lambda^3+8\lambda^2\mu+6\lambda\mu^2+\mu^3)\},$$

$$s_5 = \frac{6}{(\lambda+\eta)(\eta+\mu)^3}, \quad s_6 = -\frac{6\{\eta^2+3\lambda^2+5\lambda\mu+\mu^2+5\eta(\lambda+\mu)\}}{(\lambda+\eta)^2(\mu+\lambda)(\eta+\mu)^4},$$

$$s_7 = \frac{1}{8\eta^5(\lambda+\eta)^2(2\eta+\lambda-\mu)^3}\{44\eta^4+\eta^3(93\lambda-37\mu)+3\lambda^2(\lambda-\mu)^2 \\ +3\lambda\eta(8\lambda^2-11\lambda\mu+3\mu^2)+\eta^2(72\lambda^2-60\lambda\mu+8\mu^2)\},$$

$$s_8 = \frac{1}{8\eta^3(2\eta+\lambda-\mu)}, \quad s_9 = \frac{\{-12\eta^2+3\lambda(\mu-\lambda)+\eta(5\mu-13\lambda)\}}{16\eta^4(\lambda+\eta)(2\eta+\lambda-\mu)^2},$$

$$s_{10} = \frac{1}{\lambda(\lambda+\eta)^4(\eta+2\lambda-\mu)^3}\{\eta^3+\eta^2(13\lambda-2\mu)+\eta(38\lambda^2-20\lambda\mu+\mu^2) \\ +\lambda(32\lambda^2-30\lambda\mu+7\mu^2)\},$$

$$s_{11} = \frac{1}{\lambda^2(\lambda+\eta)^5(\eta+2\lambda-\mu)^3}\{\eta^4+2v^3(7\lambda-\mu)+\eta^2(91\lambda^2-22\lambda\mu+\mu^2) \\ +2\lambda\eta(105\lambda^2-55\lambda\mu+4\mu^2)+\lambda^2(156\lambda^2-138\lambda\mu+31\mu^2)\},$$

$$s_{12} = \frac{1}{(\lambda+\eta)^3(\eta+2\lambda-\mu)},$$

$$s_{13} = \frac{1}{(\lambda+\eta)^3(\eta+2\lambda-\mu)}\{\eta^4+2\eta^3(7\lambda-\mu)+\eta^2(91\lambda^2-22\lambda\mu+\mu^2) \\ +2\lambda\eta(105\lambda^2-55\lambda\mu+4\mu^2)+\lambda^2(156\lambda^2-138\lambda\mu+31\mu^2)\},$$

And the solution of the equation (16) is

$$u_1 = \Big\{\left(b^3+6abc\right)\left(r_1t^3+r_2t^2+r_3t+r_4\right)+(b^2c+ac^2)\left(r_5t^4+r_6t^3+r_7t^2+r_8t+r_9\right)+bc^2\left(r_{10}t^5+r_{11}t^4 \\ +r_{12}t^3+r_{13}t^2+r_{14}t+r_{15}\right)+c^3\left(r_{16}t^6+r_{17}t^5+r_{18}t^4+r_{19}t^3+r_{20}t^2+r_{21}t+r_{22}\right)\Big\}e^{-3\lambda t}+\Big\{bcd\left(r_{23}t^3 \\ +r_{24}t^2+r_{25}t+r_{26}\right)+c^2d\left(r_{27}t^4+r_{28}t^3+r_{29}t^2+r_{30}t+r_{31}\right)\Big\}e^{-(2\lambda+\mu)t}+\Big\{bch\left(r_{32}t^3+r_{33}t^2+r_{34}t+r_{35}\right) \\ +c^2h\left(r_{36}t^4+r_{37}t^3+r_{38}t^2+r_{39}t+r_{40}\right)\Big\}e^{-(2\lambda+\eta)t} \tag{25}$$

where

$$r_1 = \frac{1}{8\lambda^3(3\lambda-\eta)(3\lambda-\mu)}, \quad r_2 = \frac{3}{16\lambda^4(3\lambda-\eta)^2(3\lambda-\mu)^2}(39\lambda^2-11\lambda\eta+3\mu\eta-11\lambda\mu),$$

$$r_3 = \frac{3}{8\lambda^5(3\lambda-\eta)^3(3\lambda-\mu)^3}\{\lambda\eta(-261\lambda^2+146\lambda\mu+21\mu^2)+\eta^2(36\lambda^2-21\lambda\mu+3\mu^2) \\ +\lambda^2(459\lambda^2-261\lambda\mu+38\mu^2)\},$$

$$r_4 = \frac{3}{16\lambda^6(3\lambda-\eta)^4(3\lambda-\mu)^4}\{\lambda^3(8451\lambda^3-7263\lambda^2\mu+2127\lambda\mu^2-211\mu^3) \\ +\lambda\eta^2(2127\lambda^3-1795\lambda^2\mu+519\lambda\mu^2-51\mu^3)+\eta^3(-211\lambda^3+177\lambda^2\mu \\ -51\lambda\mu^2+5\mu^3)+\lambda^2\eta(-7263\lambda^3+6177\lambda^2\mu-1795\lambda\mu^2+177\mu^3)\},$$

$r_5 = 3r_1$, $r_6 = 0.25r_2$, $r_7 = 1.5r_3$, $r_8 = 0.75r_4$,

$$r_9 = \frac{3}{16\lambda^7(3\lambda-\eta)^5(3\lambda-\mu)^5}\{\eta^4(2059\lambda^4 - 2328\lambda^3\mu + 1014\lambda^2\mu^2 - 200\lambda\mu^3 + 15\mu^4) + \lambda\eta^3(3405\lambda^4 - 3863\lambda^3\mu + 1686\lambda^2\mu^2 - 333\lambda\mu^3 + 25\mu^4) - \eta\lambda^3(38691\lambda^4 - 44325\lambda^3\mu - 3863\lambda\mu^3 + 291\mu^4) + \lambda^2\eta^2(68355\lambda^4 - 77880\lambda^3\mu + 34082\lambda^2\mu^2 - 6744\lambda\mu^3 + 507\mu^4 + 19470\lambda^2\mu^2) + \lambda^4(268191\lambda^4 - 309528\lambda^3\mu + 136710\lambda^2\mu^2 - 27240\lambda\mu^3 + 2059\mu^4)\},$$

$$r_{10} = 3r_1,\ r_{11} = 5r_2,\ r_{12} = \frac{5}{2}r_3,\ r_{13} = 30r_4,\ r_{14} = \frac{1}{3}r_9,$$

$$r_{15} = \frac{45}{32\lambda^8(3\lambda-\eta)^6(3\lambda-\mu)^6}\{\lambda^5(3849849\lambda^5 - 5591673\lambda^4\mu + 3309930\lambda^3\mu^2 - 9221\mu^5 - 993330\lambda^2\mu^3 + 150669\lambda\mu^4) + 2\lambda^3\eta^2(1654965\lambda^5 - 2380725\lambda^4\mu + 63105\lambda\mu^4 - 3849\mu^5 + 1399482\lambda^3\mu^2 - 417778\lambda^2\mu^3) + \lambda\eta^4(150669\lambda^5 - 215501\lambda^4\mu + 126210\lambda^3\mu^2 - 37578\lambda^2\mu^3 + 5665\lambda\mu^4 - 345\mu^5) - \eta^5(9221\lambda^5 - 2290\lambda^2\mu^3 + 345\lambda\mu^4 - 21\mu^5 - 13161\lambda^4\mu + 7698\lambda^3\mu^2) - 2\eta^3\lambda^2(496665\lambda^5 - 712197\lambda^4\mu + 417778\lambda^3\mu^2 - 1145\mu^5 - 124530\lambda^2\mu^3 + 18789\lambda\mu^4) + \lambda^4\eta(5591673\lambda^5 - 8076861\lambda^4\mu + 4761450\lambda^3\mu^2 - 1424394\lambda^2\mu^3 + 215501\lambda\mu^4 - 13161\mu^5)\},$$

$$r_{16} = r_1,\ r_{17} = 2r_2,\ r_{18} = 5r_3,\ r_{19} = 20r_4,\ r_{20} = 15r_9,\ r_{21} = 2r_{15},$$

$$r_{22} = \frac{45}{16\lambda^9(3\lambda-\eta)^7(3\lambda-\mu)^7}\{\lambda^5\eta(-45099585\lambda^6 + 78771690\lambda^5\mu - 58385475\lambda^4\mu^2 + 23395932\lambda^3\mu^3 - 5329527\lambda^2\mu^4 + 653018\lambda\mu^5 - 33573\mu^6) + \lambda\eta^5(-377349\lambda^6 + 653018\lambda^5\mu - 480999\lambda^4\mu^2 - 273\mu^6 + 191868\lambda^3\mu^3 - 43555\lambda^2\mu^4 + 5322\lambda\mu^5) + \eta^6(19427\lambda^6 - 33573\lambda^5\mu + 24708\lambda^4\mu^2 - 9850\lambda^3\mu^3 + 2235\lambda^2\mu^4 - 273\lambda\mu^5 + 14\mu^6) + \lambda^2\eta^4(3074652\lambda^6 - 5329527\lambda^5\mu + 2235\mu^6 + 3929711\lambda^4\mu^2 - 1568694\lambda^3\mu^3 + 356298\lambda^2\mu^4 - 43555\lambda\mu^5) - 2\eta^3\lambda^3(6735285\lambda^6 + 4952\mu^6 - 11697966\lambda^5\mu + 863691\lambda^4\mu^2 - 3450868\lambda^3\mu^3) + \lambda^6(25711830\lambda^6 - 45099585\lambda^5\mu + 33533595\lambda^4\mu^2 + 19427\mu^6 - 13470570\lambda^3\mu^3 + 3074652\lambda^2\mu^4 - 377349\lambda\mu^5) + \lambda^4\eta^2(33533595\lambda^6 - 58385475\lambda^5\mu + 43178562\lambda^4\mu^2 - 1727382\lambda^3\mu^3 + 3929711\lambda^2\mu^4 - 480999\lambda\mu^5 + 24708\mu^6)\},$$

$$r_{25} = \frac{9}{2\lambda^3(\lambda+\mu)^5(2\lambda+\mu-\eta)^3}\{156\lambda^4 + 210\lambda^3\mu + 91\lambda^2\mu^2 + 14\lambda\mu^3 + \mu^4 + \eta^2(31\lambda^2 + 8\lambda\mu + \mu^2) + 2\eta(69\lambda^3 + 55\lambda^2\mu + 11\lambda\mu^2 + \mu^3)\},$$

$$r_{23} = \frac{3}{\lambda(\lambda+\mu)^3(2\lambda+\mu-\eta)},\ r_{24} = \frac{9}{2\lambda^2(\lambda+\mu)^4(2\lambda+\mu-\eta)^2}(16\lambda^2 + 11\lambda\mu - 7\lambda\eta - \mu\eta),$$

$$r_{26} = \frac{9}{2\lambda^4(\lambda+\mu)^6(2\lambda+\mu-\eta)^4}\{1200\lambda^6 + 2376\lambda^5\mu + 1808\lambda^4\mu^2 + 671\lambda^3\mu^3 + 135\lambda^2\mu^4 + \mu^6 + \eta^3(111\lambda^3 + 39\lambda^2\mu + 9\lambda\mu^2 + \mu^3) + \eta^2(28\lambda^4 + 645\lambda^3\mu + 189\lambda^2\mu^2 + 35\lambda\mu^3 + 3\mu^4) + \eta(1608\lambda^5 + 2296\lambda^4\mu + 1173\lambda^3\mu^2 + 285\lambda^2\mu^3 + 43\lambda\mu^4 + 3\mu^5)\},$$

$$r_{27} = 0.5r_{23},\ r_{28} = 0.67r_{24},\ r_{29} = r_{25},\ r_{30} = r_{26},$$

$$r_{31}=\frac{1}{4\lambda^5(\lambda+\mu)^7(2\lambda+\mu-\eta)^5}\{8016\lambda^8+20784\lambda^7\mu+22360\lambda^6\mu^2+13062\lambda^5\mu^3+20\lambda\mu^7$$
$$+4651\lambda^4\mu^4+1110\lambda^3\mu^5+188\lambda^2\mu^6+\mu^8+\eta^4(351\lambda^4+150\lambda^3\mu+48\lambda^2\mu^2+10\lambda\mu^3+\mu^4)$$
$$-2\eta^3(1515\lambda^5+1452\lambda^4\mu+540\lambda^3\mu^2+146\lambda^2\mu^3+25\lambda\mu^4+2\mu^5)+2\eta^2(4940\lambda^6+3\mu^6$$
$$+7385\lambda^5\mu+4263\lambda^4\mu^2+1370\lambda^3\mu^3+314\lambda^2\mu^4+45\lambda\mu^5)-2\eta(7224\lambda^7+14728\lambda^6\mu$$
$$+12049\lambda^5\mu^2+5272\lambda^4\mu^3+1460\lambda^3\mu^4+386\lambda^2\mu^5+35\lambda\mu^6+2\mu^7)\},$$

$$r_{32}=\frac{3}{\lambda(\lambda+\eta)^3(2\lambda+\eta-\mu)},\ r_{33}=\frac{9}{2\lambda^2(\lambda+\eta)^4(2\lambda+\eta-\mu)^2}(16\lambda^2+11\lambda\eta+\eta^2-7\lambda\mu-\mu\eta),$$

$$r_{34}=\frac{9}{2\lambda^3(\lambda+\eta)^5(2\lambda+\eta-\mu)^3}\{\eta^4+2\eta^3(7\lambda-\mu)+\eta^2(91\lambda^2-22\lambda\mu+\mu^2)$$
$$+2\lambda\eta(105\lambda^2-55\lambda\mu+4\mu^2)+\lambda^2(156\lambda^2-138\lambda\mu+31\mu^2)\},$$

$$r_{35}=\frac{9}{4\lambda^4(\lambda+\eta)^6(2\lambda+\eta-\mu)^4}\{\eta^6+\eta^5(17\lambda-3\mu)+\eta^4(135\lambda^2-43\lambda\mu+3\mu^2)+\lambda^3$$
$$(1200\lambda^3-1608\lambda^2\mu+728\lambda\mu^2-111\mu^3)+\lambda^2\eta(2376\lambda^3-2296\lambda^2\mu+645\lambda\mu^2-39\mu^3)$$
$$+\lambda\eta^2(1808\lambda^3-1173\lambda^2\mu+189\lambda\mu^2-9\mu^3)+\eta^3(671\lambda^3-285\lambda^2\mu+35\lambda\mu^2-\mu^3)\},$$

$$r_{36}=0.5r_{32},\ r_{37}=0.67r_{33},\ r_{38}=r_{34},\ r_{39}=2r_{35},$$

$$r_{40}=\frac{9}{4\lambda^5(\lambda+\eta)^7(2\lambda+\eta-\mu)^5}\{\eta^8+4\eta^7(5\lambda-\mu)+2\eta^6(94\lambda^2-35\lambda\mu+3\mu^2)+2\eta^5(555\lambda^3$$
$$-286\lambda^2\mu+45\lambda\mu^2-2\mu^3)+\eta^4(4651\lambda^4-2920\lambda^3\mu+628\lambda^2\mu^2-50\lambda\mu^3+\mu^4)$$
$$+2\lambda\eta^3(6531\lambda^4-5272\lambda^3\mu+1370\lambda^2\mu^2-146\lambda\mu^3+5\mu^4)+2\lambda^2\eta^2(11180\lambda^4$$
$$-12049\lambda^3\mu+4263\lambda^2\mu^2-540\lambda\mu^3+24\mu^4)+2\lambda^3\eta(10392\lambda^4-14728\lambda^3\mu+75\mu^4$$
$$+7385\lambda^2\mu^2-1452\lambda\mu^3)+\lambda^4(8016\lambda^4-14448\lambda^3\mu+9880\lambda^2\mu^2-3030\lambda\mu^3+351\mu^4)\},$$

Substituting the values of A_1, B_1, C_1, D_1 and H_1 from equations (22), (21), (2), (23) and (24), into the equation (4), we obtain

$$\dot{a}=\varepsilon\left(p_1a^2b+p_2ab^2+p_3a^3+p_4a^2c\right)e^{-2\lambda t}$$

$$\dot{b}=\varepsilon\Big\{\left(m_1a^2b+m_2ab^2+m_3a^2c\right)e^{-2\lambda t}+\left(m_4abh+m_5b^2h+m_6ach\right)e^{-(\lambda+\eta)t}$$
$$+\left(m_7bd^2+m_8cd^2\right)e^{-2\mu t}+\left(m_9bdh+m_{10}cdh\right)e^{-(\mu+\eta)t}+\Big(m_{11}abd+m_{12}b^2d$$
$$+m_{13}acd\Big)e^{-(\lambda+\mu)t}+\left(m_{14}bh^2+m_{15}ch^2\right)e^{-2\eta t}\Big\}$$

$$\dot{c}=\varepsilon\Big\{\left(l_1ab^2+l_2a^2c\right)e^{-2\lambda t}+\left(l_3acd+l_4b^2d\right)e^{-(\lambda+\mu)t}+\left(l_5ach+l_6b^2h\right)e^{-(\lambda+\eta)t}$$
$$+l_7cd^2e^{-2\mu t}+l_8ch^2e^{-2\eta t}+l_9cdhe^{-(\mu+\eta)t}\Big\} \tag{26}$$

$$\dot{d}=\varepsilon\left\{q_1d^3e^{-2\mu t}+\left(q_2ad^2+q_3bd^2+q_4cd^2\right)e^{-(\lambda+\mu)t}+\left(q_5abd+q_6b^2d+q_7a^2d+q_8acd\right)e^{-2\lambda t}\right\}$$

$$\dot{h}=\varepsilon\Big\{s_1h^3e^{-2\eta t}+s_2dh^2e^{-(\mu+\eta)t}+s_3d^2he^{-2\mu t}+\left(s_4cdh+s_5adh+s_6bdh\right)e^{-(\lambda+\mu)t}$$
$$+\left(s_7ch^2+s_8ah^2+s_9bh^2\right)e^{-(\lambda+\eta)t}+\left(s_{10}abh+s_{11}b^2h+s_{12}a^2h+s_{13}ach\right)e^{-2\lambda t}\Big\}$$

Here, all of the equations of (26) have no exact solutions. However, since $\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{h}$ are proportional to the small parameter

ε, they are gradually varying functions of time t. Thus, it is feasible to substitute a, b, c, d, h by their respective values obtained in linear case (*i.e.*, the values of a, b, c, d, h obtained when $\varepsilon = 0$) in the right hand side of equation (26). This type of substitution was first introduced by Murty *et al*. [12], and Murty and Deekshatulu [5] in order to solve similar type of nonlinear equations. The solution of (26), thus, become

$$
\begin{aligned}
a &= a_0 + \varepsilon\left(p_1 a_0^2 b_0 + p_2 a_0 b_0^2 + p_3 a_0^3 + p_4 a_0^2 c_0\right)\frac{1-e^{-2\lambda t}}{2\lambda} \\
b &= b_0 + \varepsilon\Big\{\left(m_1 a_0^2 b_0 + m_2 a_0 b_0^2 + m_3 a_0^2 c_0\right)\frac{1-e^{-2\lambda t}}{2\lambda} + \left(m_4 a_0 b_0 h_0 + m_5 b_0^2 h_0 + m_6 a_0 c_0 h_0\right) \\
&\quad \frac{1-e^{-(\lambda+\eta)t}}{\lambda+\eta} + \left(m_7 b_0 d_0^2 + m_8 c_0 d_0^2\right)\frac{1-e^{-2\mu t}}{2\mu} + \left(m_9 b_0 d_0 h_0 + m_{10} c_0 d_0 h_0\right)\frac{1-e^{-(\mu+\eta)t}}{\mu+\eta} \\
&\quad + \left(m_{11} a_0 b_0 d_0 + m_{12} b_0^2 d_0 + m_{13} a_0 c_0 d_0\right)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + \left(m_{14} b_0 h_0^2 + m_{15} c_0 h_0^2\right)\frac{1-e^{-2\eta t}}{2\eta}\Big\} \\
c &= c_0 + \varepsilon\Big\{\left(l_1 a_0 b_0^2 + l_2 a_0^2 c_0\right)\frac{1-e^{-2\lambda t}}{2\lambda} + \left(l_3 a_0 c_0 d_0 + l_4 b_0^2 d_0\right)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + \big(l_5 a_0 c_0 h_0 \\
&\quad + l_6 b_0^2 h_0\big)\frac{1-e^{-(\lambda+\eta)t}}{\lambda+\eta} + l_7 c_0 d_0^2\frac{1-e^{-2\mu t}}{2\mu} + l_8 c_0 h_0^2\frac{1-e^{-2\eta t}}{2\eta} + l_9 c_0 d_0 h_0\frac{1-e^{-(\mu+\eta)t}}{\mu+\eta}\Big\} \\
d &= d_0 + \varepsilon\Big\{q_1 d_0^3\frac{1-e^{-2\mu t}}{2\mu} + \left(q_2 a_0 d_0^2 + q_3 b_0 d_0^2 + q_4 c_0 d_0^2\right)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} \\
&\quad + \left(q_5 a_0 b_0 d_0 + q_6 b_0^2 d_0 + q_7 a_0^2 d_0 + q_8 a_0 c_0 d_0\right)\frac{1-e^{-2\lambda t}}{2\lambda}\Big\} \\
h &= h_0 + \varepsilon\Big\{s_1 h_0^3\frac{1-e^{-2\eta t}}{2\eta} + s_2 d_0 h_0^2\frac{1-e^{-(\mu+\eta)t}}{\mu+\eta} + s_3 d_0^2 h_0\frac{1-e^{-2\mu t}}{2\mu} + \big(s_4 c_0 d_0 h_0 \\
&\quad + s_5 a_0 d_0 h_0 + s_6 b_0 d_0 h_0\big)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + \left(s_7 c_0 h_0^2 + s_8 a_0 h_0^2 + s_9 b_0 h_0^2\right)\frac{1-e^{-(\lambda+\eta)t}}{\lambda+\eta} \\
&\quad + \left(s_{10} a_0 b_0 h_0 + s_{11} b_0^2 h_0 + s_{12} a_0^2 h_0 + s_{13} a_0 c_0 h_0\right)\frac{1-e^{-2\lambda t}}{2\lambda}\Big\}
\end{aligned}
\tag{27}
$$

Hence, we obtain the first approximate solution of the equation (14) as

$$x(t,\varepsilon) = (a + bt + ct^2)e^{-\lambda t} + de^{-\mu t} + he^{-\eta t} + \varepsilon u_1 \tag{28}$$

where a, b, c, d, h are given by the equation (27) and u_1 is given by (25).

4. Results and Discussion

An approximate solution of fifth order more critically time dependent damped nonlinear system with constant has been obtained based on the KBM method. The solution can theoretically be obtained up to the accuracy of any order of approximation. However, it is mathematically difficult to find out a more accurate solution due to some algebraic intricacy for the derivation. We compare the approximate solution to the numerical solution (considered to be exact) with a view to bring the efficiency of an approximate solution obtained by a certain perturbation method. Here, $x(t, \varepsilon)$ is computed by equation (28), where a, b, c, d, h are calculated from equation (27); and (25) is used to obtain u_1 when $\varepsilon = 0.01$, along with the different sets of initial conditions. The results are represented in Figure 1 to Figure 5. It is apparent that, for both sets of initial conditions, our perturbation results display good coincidence with the numerical results.

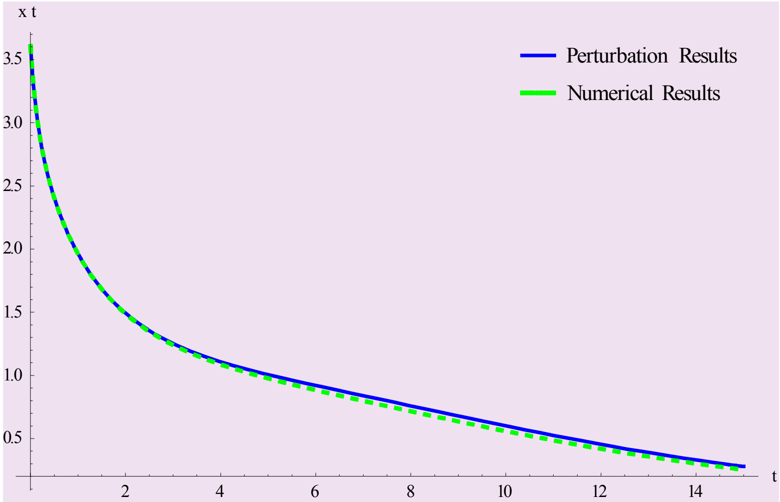

Figure 1. Comparison between perturbation and numerical results for $\lambda = 0.3$, $\mu = 2.1$, $\eta = 9$ and $\varepsilon = 0.01$ with the initial conditions $a_0 = 0.35$, $b_0 = 0.15$, $c_0 = 0.1$, $d_0 = 0.42$, $h_0 = 0.45$.

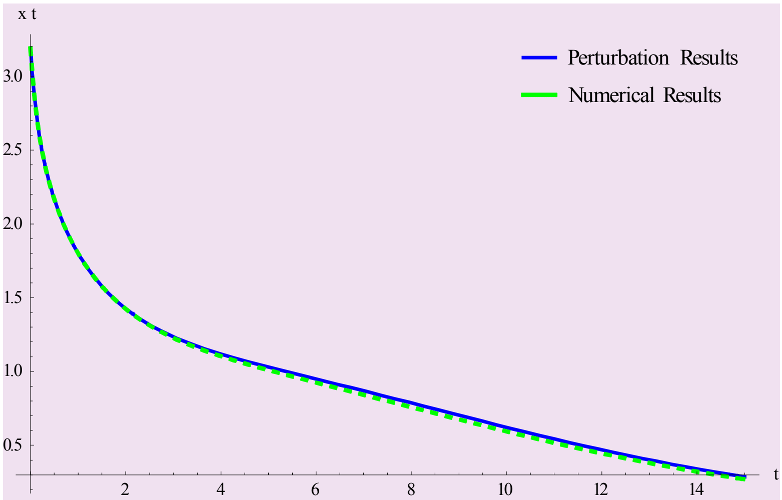

Figure 2. Comparison between perturbation and numerical results for $\lambda = 0.3$, $\mu = 2.4$, $\eta = 8.4$ and $\varepsilon = 0.01$ with the initial conditions $a_0 = 0.3$, $b_0 = 0.2$, $c_0 = 0.05$, $d_0 = 0.45$, $h_0 = 0.35$.

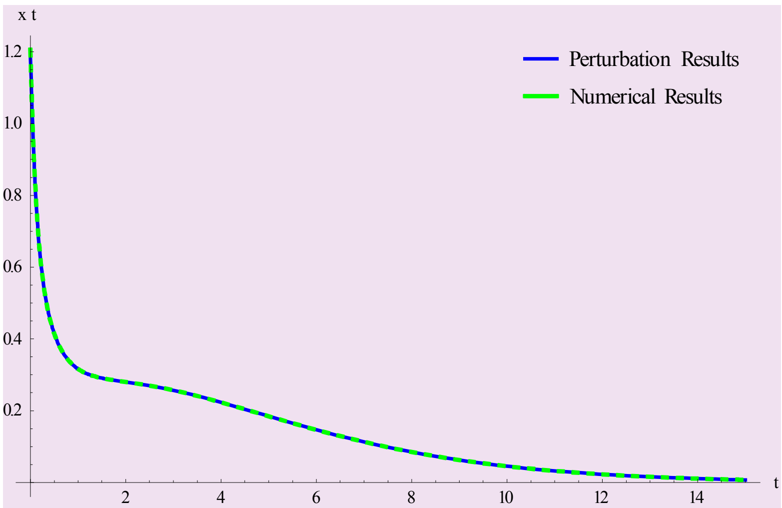

Figure 3. Comparison between perturbation and numerical results for $\lambda = 0.5$, $\mu = 2.5$, $\eta = 9$ and $\varepsilon = 0.01$ with the initial condition $a_0 = 0.25$, $b_0 = 0.15$, $c_0 = 0.05$, $d_0 = 0.5$, $h_0 = 0.45$.

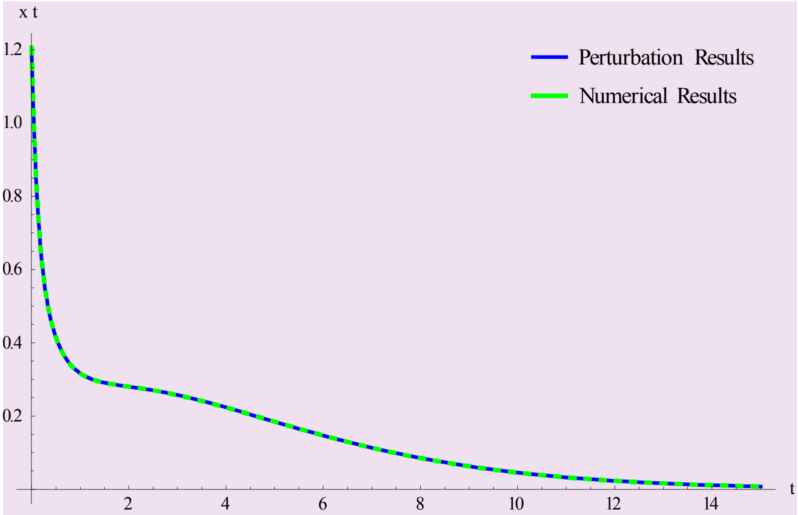

Figure 4. Comparison between perturbation and numerical results for $\lambda = 0.7$, $\mu = 2.9$, $\eta = 9.5$ and $\varepsilon = 0.01$ with the initial conditions $a_0 = 0.3$, $b_0 = 0.15$, $c_0 = 0.1$, $d_0 = 0.42$, $h_0 = 0.45$.

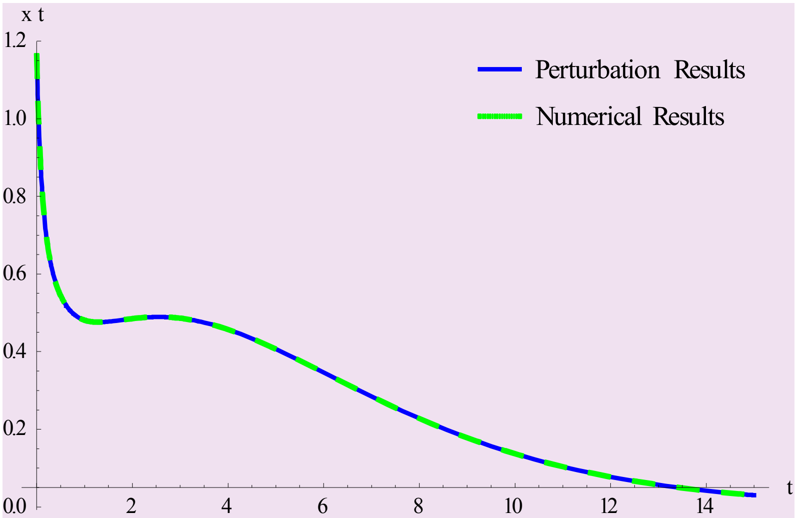

Figure 5. Comparison between perturbation and numerical results for $\lambda = 0.45$, $\mu = 2.4$, $\eta = 11$ and $\varepsilon = 0.01$ with the initial conditions $a_0 = 0.35$, $b_0 = 0.20$, $c_0 = 0.1$, $d_0 = 0.40$, $h_0 = 0.35$.

5. Conclusion

In this paper, we modify and successfully apply the Krylov-Bogoliubov-Mitropolskii (KBM) method, which is also known as the most widely used perturbation method for studying the transient behavior of nonlinear systems, to the fifth order more critically damped nonlinear systems. All the solutions here are obtained in such circumstances, where the three eigenvalues are equal and other two are distinct with respect to the fifth order more critically damped nonlinear systems. It should be mentioned here that, generally, inaccuracy occurs in the KBM method due to rapid changes of x with respect to time t. However, in relation to the different sets of initial conditions, all the above figures in the time period $t = 0$ to $t = 15$ suggest that the perturbation solutions obtained by the modified KBM method correspond exactly to the numerical solutions. Further, it should be mentioned here that, all the calculations and results in this article have been computed through *Mathematica 9.0*. Finally, the modified KBM method reveals highly accurate results that may be applied for different kinds of nonlinear differential systems where the small nonlinearity is present. The method is, thus, not contingent on whether or not the system has eigenvalues real, or complex conjugate, or pure imaginary. It is, therefore, concluded that the method is not dependent on the order of the system.

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