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Golden Mean and the Action of Mobius Group M

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Abstract

Golden mean is an important feature of Aristotle's' virtue theory consequently it is significant to understand and it should be easily applied to any problem. The abilities that surround Aristotle's ethics are to be found within the Golden mean, which includes finding the balance among two means. The Golden mean, the Golden section and Golden ratio all the admired names for a mathematical concept which is defined as a number that is equal to its own reciprocal plus one. Suppose we have the extended complex plane and we define Möbius group which has order six and it is defined by linear functional transformations from extended complex plane to extended complex plane. In this paper we determine the existence of golden mean in the action on M real quadratic field and we give the orbit in which golden mean appears.

Keywords

Golden Mean, Pisot Number, Salem Number, Orbit, Mobius Group

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1. Introduction

Quadratic fields are a basic object of study of examples in algebraic number theory. The study of groups by means of their actions on several sets and algebraic structures has developed into a valuable technique group actions on fields have various applications in physics, symmetries, algebraic geometry and cryptology. Congruence is nothing more than a statement of divisibility but it often helps to discover proofs and suggests new ideas to resolve problems.

Mobius groups have always been of powerful tool in finding group actions over quadratic fields. The Theory of congruence was familiarized by Card Friedrich Gauss, one of the greatest Mathematicians of all times. The Congruence is not anything more than a statement of divisibility. Though, it often helps to discover proofs and we realize that congruence advocates novel ideas to resolve the problems that will lead us to advance inspiring beliefs. We have utilize congruence classes in order to explore the action of Möbius groups on the real quadratic fields. Therefore these congruence classes have been widely used to compute the action of Möbius groups on real and imaginary quadratic fields as well.

It is well known that every real quadratic irrational number $Q(\sqrt{m}\,)$can be written uniquely as $\frac{a+\sqrt{n}}{c}$ where n is a non-square positive integer and $a,\frac{(a^2-n)}{c},c$ are relatively prime integers. Let $n = k^2m$, where m a square free positive integer and $k > 0$ be an integer, then

$$Q^*(\sqrt{n}\,):=\left\{\frac{a+\sqrt{n}}{c}:a,0b:=\frac{a^2-n}{c},0\ c \in Z \text{ and } \left(a,\frac{a^2-n}{c},c\right)=1\right\}$$

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is a proper G − subset of $Q(\sqrt{m})$ for all k.

Our interest is to discover linear transformation in general x, y satisfying the relations $x^2 = y^m = 1$, when studying an action of the group $\langle x, y \rangle$ on real quadratic fields. The group $\langle x, y \rangle$ is trivial when $m = 1$, for $m = 2$, it is an infinite dihedral group and does not give inspiring information while studying its action on the real quadratic irrational numbers. If $m = 3$, the it is a modular group $PSL(2, Z)$. Here we are focused towards group $\langle x, y \rangle$ for $m = 6$. That is $M = \langle x, y;\ x^2 = y^6 = 1 \rangle$.

Some proper subgroups of the Mobius group M has been investigated in [9] and particularly an important subgroup M' has been defined to prove that $Q^*(\sqrt{n})$ is invariant under the action of M′.

A classification of the real quadratic irrational numbers $\frac{a+\sqrt{n}}{c}$ of $Q^*(\sqrt{n})$ with respect to modulo 3^r has been introduced. Using this formula, the cardinality of the set $I^{n}_{3^r}$ with n modulo 3^r can be calculated. These equivalence classes play a vital part in the construction of G and M subsets of $Q(\sqrt{m})$. It is similarly a very beneficial technique to compute the orbits of certain invariant subsets of real quadratic fields under the action of Mobius groups [5].

Actions of Mobius groups M and M′ on $Q(\sqrt{m})$ is presented [7]. The system of linear congruence has been introduced to discover classes [a, b, c](mod 12) of elements of $Q^*(\sqrt{n})$. Then by means of these classes, several M′ − subsets has been found which assist in finding more M − subsets of $Q(\sqrt{m})$. A relationship among the actions of group G and M on $Q(\sqrt{m})$ has been established in [6]. It characterized several significant elements of G in terms of generators of M and vice versa. Therefore a correlation between the rudiments of these two Modular Groups helps in discovering various G and M − subsets of $Q(\sqrt{m})$.

In [8] we have validated certain significant properties related to symmetries of continued fraction in quadratic irrationals which help to understand interesting results. Precisely, we have established a key result that infers that the set of quadratic irrationals be able to be characterize as the set of numbers with periodic continued fraction to set down the connection between symmetry in the continued fraction expansion of the quadratic irrational. The origin of continued fraction is conventionally retained at the time of Euclid's Algorithm; however the concept of continued fraction is used to compute the greatest common divisor of two numbers.

Let G be group acting on a set X. Let $\lambda \in X$ and define $\lambda^G = \{\beta \in X: \beta = g(\lambda), g \in G\}$. Then λ^G is called G- orbit of α in X. In [2] the natural action of some subgroups of G and M on the elements of quadratic number field over rational number has been discussed. The authors have considered sets of numbers with fixed discriminant in the quadratic field. Then the structure of orbits of the action of G, M and $G \cap M$ is observed.

A pisot number is a positive algebraic integer greater than 1, all of whose conjugate elements have absolute value less than 1. A real quadratic algebraic integer greater than 1 and of degree 2 or 3 is a pisot number if its norm is equal to ± 1. The golden mean $\varphi = \frac{1+\sqrt{5}}{2}$ is an example of a pisot number, since it has degree 2 and norm -1 (denoted by φ_0 when considered as a piostnumber). An algebraic integer λ> 1 a Salem number if its conjugate satisfies $|\acute{\lambda}| < 1$. The golden mean, the golden section and golden ratio all the well-liked names for a mathematical concept which is defined as that which is equal to its own reciprocal plus one, multiplying both sides of this equation by the golden ratio. It is known that every Pisot number is a limit of Salem numbers. At present there are 47 known Salem numbers less than 1.3 and the list is known to be complete through degree 40. There is a well-known relationship between coveter systems, Salem numbers, and golden mean. We know that the Pisot numbers from a closed subset P ⊂ R, where R is a field of real numbers. The smallest Pisot number λ_p, equivalent to 1.324717, is a root of $x^3 - x - 1 = 0$, while the smallest accumulation point in P is the Golden mean, $\lambda_M = \frac{1+\sqrt{5}}{2}$ equivalent to 1.61803. Note that $\lambda_M^2 = \frac{3+\sqrt{5}}{2}$ is equivalent to 2.61803….

2. Golden Mean and M

We derive the appealing property that the square of the golden ratio is equal to simple number itself plus one. Since that equation can be written as φ^2-φ-1 =0, we can derive the value of golden ratio from the quadratic equation. The number represented by the Greek letter φ (phi) is irrational i.e. $\varphi = \frac{1+\sqrt{5}}{2}$.

Theorem:

In an action of Möbius Group on $Q(\sqrt{5}) \cup \{\infty\}$, λ_M is the fixed point of the commutor of the Möbius Group $M = \langle t, u: t^2 = u^6 = 1 \rangle$.

Proof: It is well known that the Mobius Group is generated by the linear fractional transformation

$T(\alpha) = \frac{-1}{3\alpha}$ & $U(\alpha) = -\frac{1}{3(\alpha+1)}$ which obviously satisfy the relation $\langle T, U : T^2 = U^6 = 1 \rangle$. Therefore, $\lambda_M T = \frac{-1+\sqrt{5}}{-6}$, $\lambda_M Tu = \frac{3+\sqrt{5}}{2}$, $\lambda_M Tu^2 = \frac{3+\sqrt{5}}{6}$, $\lambda_M Tu^3 = \frac{-1+\sqrt{5}}{2}$, $\lambda_M Tu^3 T = \frac{1+\sqrt{5}}{-6}$, $\lambda_M Tu^3 Tu = \frac{5+\sqrt{5}}{-10}$, $\lambda_M Tu^3 Tu^2 = \frac{5+\sqrt{5}}{-6}$, $\lambda_M Tu^3 Tu^3 = \frac{-3+\sqrt{5}}{6}$, $\lambda_M Tu^2 Tu^2 T = \frac{-5+\sqrt{5}}{-10}$, $\lambda_M Tu^3 Tu^2 Tu = \frac{-1+\sqrt{5}}{-6}$, $\lambda_M Tu^3 Tu^2 Tu^2 = \frac{3+\sqrt{5}}{2}$, $\lambda_M Tu^3 Tu^2 T uT = \lambda_M = \frac{1+\sqrt{5}}{2}$.

Corollary:

The minimal polynomial of Golden Mean is $\lambda_M^2 - \lambda_M - 1 = 0$.

Proof:

$$u^3 tu^5 t\lambda_M = u^3 t u^4 \left(\frac{-(\lambda_M^2+2)}{11}\right) = u^3 tu^3 \left(\frac{(\lambda_M+7)}{-15}\right) = u^3 tu^2 \left(\frac{-5}{8-\lambda_M}\right) = u^3 t(\lambda_M - 1) = \left(\frac{-1}{(\lambda_M'+2)}\right) = u\left(\frac{-5}{3(\lambda_M'+2)}\right) = -\frac{1}{\lambda_M+1} + 2 = \lambda_M$$

Therefore, $\lambda_M tu^5 tu^3 = \lambda_M$, *and so* $-\frac{1}{\lambda_M+1} + 2 = \lambda_M$, yields $\lambda_M^2 - \lambda_M - 1 = 0$ which is required polynomial of the Golden Mean.

Corollary:

Let λ_M' denote the algebraic conjugate of λ_M, then

(i) $t\lambda_M = -\frac{1}{3}\lambda'_M$, $ut\lambda_M = \frac{-(\lambda_M+3)}{11}$, $u^2 t\lambda_M = \frac{(-7\lambda_M'+1)}{15\lambda_M'}$, $u^3 t\lambda_M = \frac{-5}{(\lambda_M'+7)}$, $u^4 t\lambda_M = -1 - \frac{\lambda_M}{3}$, $u^5 t\lambda_M = -\lambda_M'$

(ii) $((t(u^5 t(\lambda_M)) = \frac{-\lambda_M}{3}$, $((u(t(u^5 t(\lambda_M)) = \frac{(\lambda_M+2)}{-5}$, $((u^2(t(u^5 t(\lambda_M)) = \frac{(-\lambda_M-2)}{3}$, $((u^3(t(u^5(t(\lambda_M)) = \lambda_M$.

Proof:

The proof follows from Corollary given above.

All Pisot numbers $\lambda, \lambda_M + \in$ are known. The Salem numbers are less than well understood. The catalog of 39 Salem numbers given in [2] include all Salem numbers λ<1.3 of degree less than or equal to 20 over the field of rational. At present there are 47 known Salem numbers λ<1.3, and the list of such is to be complete through degree 40 [5].

3. Approximation of the Golden Mean

In this section we give approximation of the golden mean. The golden mean $\lambda_M = \frac{1+\sqrt{5}}{2}$ is the quadratic irrationality, which is hardest to approximate by rational numbers, that is $\lambda_M - \frac{p}{q} \neq 0$, where p and q are co- prime integers. We make $\left|\lambda_M - \frac{p}{q}\right|$ as small as possible for a fixed q, i.e., $\left|\lambda_M - \frac{p}{q}\right| < \epsilon_q(\lambda_M)$, when $\epsilon_q(\lambda_M)$ tends to zero as q tends to infinity. Trivially, $\epsilon_q(\lambda_M) < \frac{1}{2q}$ We can, in fact, for any irrational α, choose a sequence $q_1, q_2, q_3, \ldots, q_n, \ldots$, tending to infinity such that $\epsilon_{q_i}(\alpha) < \frac{1}{q_i^2}$. For the number $\lambda_M = \frac{1+\sqrt{5}}{2}$, we can not do better than this. The orbit in which golden mean appears is given in figure 1.

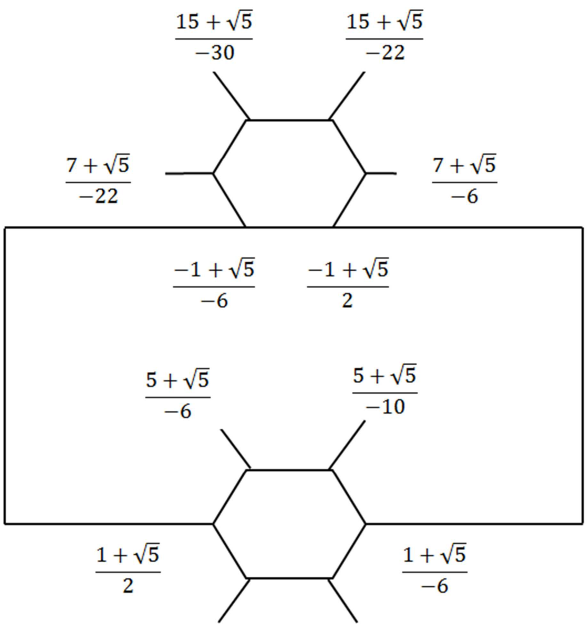

Figure 1. Orbit of Golden mean.

4. Conclusion

The Golden mean is a vital facet of Aristotle's' virtue theory therefore it is significant that it is understood and can be easily applied to any question. In current manuscript, we determine the presence of golden mean in the action of Mobius group M on real quadratic field and we contribute the orbit in which golden mean has appeared.

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