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Total Variation Distance Between Poisson Distribution and Polya Distribution and It's Non-uniform Upper Bound

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Abstract

The approximation of distribution due to Egege et al [1] is further extended to a wider approximation called total variation distances between Poisson and Pólya distribution, using Stein's Chen method and *ω*-function to give a non-uniform bound. In this work it was found that the results obtain for non-uniform are better than the results obtained for uniform bound in literature. Polya distribution approximate Poisson sufficiently enough provided *r* is close to *n* and *N* is large. If the upper bound is very small then a good approximation is obtained.

Keywords

Variation Distance, Poisson Distribution, Pólya Distribution, Stein Method, Non-uniform Upper Bound

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1. Introduction

Pólya distribution was first studied and presented by G. Pólya 1923. Polyadistribution can be explained in a better way with the use of urn scheme. Suppose that we have an urn that initially contains a red and green balls denoted by r and $b$respectively, where r and b are positive integers. At each discrete time$(trial)$, a ball is selected from the urn and then return the ballot the urn along with$c$, new balls of the same color. It is a discrete distribution that depend on four parameters (r, n, p, c). Anon –negative integer –valued random variable X taking non-negative integer values k, $0 \leq k \leq n$. It's probability distribution function is of the form

$$P_X(k) = \binom{n}{k} \frac{(p;\gamma)_{k-1}(q;\gamma)_{n-k-1}}{(1;\gamma)_{n-1}}, \quad (1)$$

Where $n > 0, 0 < p < 1, q = 1 - p$and $\gamma > 0$.

The form (1) can also be expressed as

$$\frac{\binom{\left(\frac{p}{\gamma}\right)+k-1}{k}\binom{\left(\frac{q}{\gamma}\right)+n-k-1}{n-k}}{\binom{\left(\frac{1}{\gamma}\right)+n-1}{n}}, k = 0,1,2 \ldots n. \quad (2)$$

The mean and variance are given as np and $\frac{np(1+\gamma n)}{(1+\gamma)}$ respectively.

Equation (2) is equivalent to Egege, et al [1], if for a specialcases $c = 1, p = \frac{r}{r+b} and \gamma = \frac{c}{r+b}$. From Equation (2) one gets;

$$P_X(k) = \binom{n}{k} \frac{\left[\frac{p}{\gamma} \ldots\ldots \frac{p}{\gamma}+(k-1)\right]\left[\frac{q}{\gamma} \ldots\ldots \frac{q}{\gamma}+(n-k-1)\right]}{\left[\frac{N}{\gamma} \ldots \frac{N}{\gamma}+(n-1)\right]} = \binom{n}{k} \frac{\left[\frac{p}{\gamma} \ldots \frac{p}{\gamma}+\tau\right]\left[\frac{q}{\gamma} \ldots \frac{q}{\gamma}+\delta\right]}{\frac{N}{\gamma} \ldots\ldots \frac{N}{\gamma}+(n-1)},$$

where $\tau = \begin{cases} 0 if k = 0 \\ k - 1 if k = 1,2,..n \end{cases}$, and

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$$\delta = \begin{cases} 0 if k = 0,1,..n \\ n-k-1\ if k = 0 \end{cases}$$

$$= \binom{n}{k} \frac{\left[N\left(\frac{r}{N}\ldots\ldots\frac{r}{N}+\frac{\tau}{N^2\gamma}\right)\right]\left[N\left(\frac{b}{N}\ldots\frac{b}{N}+\frac{\delta}{N^2\gamma}\right)\right]}{N\left[\frac{1}{N^2\gamma}\ldots..1+\frac{(n-1)}{N^2\gamma}\right]}$$

$$= \binom{n}{k} \frac{\left[N^k\left(\frac{r}{N}\ldots\frac{r}{N}+\frac{\tau}{N^2\gamma}\right)\right]\left[N^{n-k}\left(\frac{b}{N}\ldots\frac{b}{N}+\frac{\delta}{N^2\gamma}\right)\right]}{N^n\left[\frac{1}{N^2\gamma}\ldots1+\frac{n-1}{N^2\gamma}\right]}.$$

Note $\frac{N^k.N^{n-k}}{N^n} = N^{n-n} = 1$ and

$$P_X(k) = \binom{n}{k} \frac{\left[\frac{r}{N}\ldots\frac{r}{N}+\frac{\tau}{N^2\gamma}\right]\left[\frac{b}{N}\ldots\frac{b}{N}+\frac{\delta}{N^2\gamma}\right]}{\left[1\ldots1+\frac{n-1}{N^2\gamma}\right]}. \quad (3)$$

If $b \to \infty, r \to \infty$ and $\gamma \to 0$ while $\frac{r}{N}$ remain fixed implies $\frac{b}{N}$ is also fixed then

$$P_X(k) = \binom{n}{k} \frac{\left[\frac{r}{N}\ldots\frac{r}{N}+\frac{\tau}{N^2\gamma}\right]\left[\frac{b}{N}\ldots\frac{b}{N}+\frac{\delta}{N^2\gamma}\right]}{\left[1\ldots1+\frac{n-1}{N^2\gamma}\right]} \to \binom{n}{k}\left(\frac{r}{N}\right)^n\left(\frac{b}{N}\right)^{n-k}. \quad (4)$$

In view of (4) Teerapabolarn [2] used Stein's Chen method [3] and ω-function associated with Polya random variable to give a bound for total variation distance between Binomial and Pólya of the form

$$d\left(B(n,p), P_{\gamma}(N,n,r,c)\right) \leq \frac{(1-p^{n+1}-q^{n+1})c(n-1)n}{(n+1)(N+c)}, \quad (5)$$

Where N, n, r, c are non-negative valued integers, $B(n,p)$ and $P_{\gamma}(N,n,r,c)$ are binomial and Polya distribution respectively.

Also in view of (4) Egege, et al [1] use Poisson to approximate Polya distribution in terms of point metric to obtain a uniform upper bound given as

$$|P_{\chi}(x) - \wp_\lambda(x)| \leq \left(1-e^{-\lambda}\right)\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}, \quad (6)$$

where $x\epsilon$ ℕ, $\wp_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $\lambda = \frac{nr}{r+b}$ and $(n-1)c \leq r$.

It is observed that the upper bound in (6) can be seen to uniform bound with respect to x_0 and do not depend on x_0. Hence the uniform upper bound in(6) may not be sufficiently good for measuring the accuracy of the approximation. To overcome these limitations, we proposed a non-uniform upper bound that depends on x_0 of the form.

$$d_{tv}\left(P_X(A), \wp_\lambda(A)\right) \leq \lambda^{-1}\left(\lambda + e^{-\lambda} - 1\right)\frac{r(N+N\gamma)-b(n-1)N\gamma}{N(N+N\gamma)} \text{ for } x_0 = 0, \quad (7)$$

and

$$d_{tv}\left(P_X(A), \wp_\lambda(A)\right) \leq min\left\{1-e^{-\lambda}, \frac{\lambda}{x_0}\right\}\frac{r(N+N\gamma)-b(n-1)N\gamma}{N(N+N\gamma)} \text{ for } x_0 > 0. \quad (8)$$

2. Method

The tools for giving the desired result consist of Stein's Chen for Poisson distribution method and ω-function associated with Pólya random variable.

2.1. Stein-Chen Method

The Stein- Chen method is a general method in probability theory to obtain bounds on distance between two probability distributions with respect to a probability metric. This method was first introduced in [4] to obtain a bound between the distributions, a sum of $m-$ dependent sequence of random variable and standard normal distribution in the Kolmogorov (uniform) metric. Stein's Chen gave an equation for Poisson distribution with parameter $\lambda > 0$ and for a given h then is given as

$$h(x) + \wp_\lambda(x) = \lambda g(x+1) - xg(x) \quad (9)$$

where $\wp_\lambda(h) = \sum_{i=0}^{\infty} h(l)\frac{e^{-\lambda}\lambda^l}{l!}$ and g, h are bounded real-valued function defined on ℕ ∪ {0}.

For $A \subseteq$ ℕ $U\{0\}$, define an indicator function $h_A(x) =$ ℕ $U(0) \to$ ℝ such that

$$h_A(x) = \begin{cases} 1 if x\epsilon\ \Omega \\ 0 if x \notin \Omega' \end{cases} \quad (10)$$

for$A = \{x_0\}$ with$x_0\ \epsilon$ ℕ $U\{0\}$. Letting $g = g_A$ and $h = h_{A.}$. Barbour, et al [5] obtained a solution of (9) expressed in the form.

$$f_A(x) = \begin{cases} (x-1)!\,\lambda^{-x}e^{\lambda}\left[\wp_\lambda(h_\Omega)\wp_\lambda\left(1-h_{C_{x-1}}\right)\right] if x > A, \\ 0\ otherwise \end{cases} \quad (11)$$

Where$C_x = (0,1,\ldots.n)$.

Lemma 2.1: let $A \in$ ℕ ∪ {0}, $x \in$ ℕ defining a non-increasingfunction as

$\Delta g_A(x) = g_A(x+1) - g_A(x)\ \forall\ x \in$ ℕ then following holds;

$$|\Delta g_A(x)| \leq \lambda^{-1}\left(1-e^{-\lambda}\right). \quad (12)$$

Proof

For $A \subseteq$ ℕ ∪ {0} and $x_0$then $x > x_0$.

By definition of a function $\Delta g_A(x) = g_A(x+1) - g_A(x)\ \forall\, x \in \mathbb{N}$

$$|\Delta g_A(x)| = |g_A(x+1) - g_A(x)| \leq |g_A(x)|$$

And lemma(2.1)gives$|g_A(x)| = \lambda^{-1}(1-e^{-\lambda})$. Thus

$$|\Delta g_A(x)| \leq \lambda^{-1}(1-e^{-\lambda}). \qquad (13)$$

Lemma 2.2: let $x_0 \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{N}$then the following hold;

1. For $x_0 = 0$

$$|\Delta g_A(x)| \leq \lambda^{-2}(\lambda + e^{-\lambda} - 1). \qquad (14)$$

2. For $x_0 > 0$

$$|\Delta g_A(x)| \leq \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}\right\}. \qquad (15)$$

Proof(15)

$$g_A(1) = (x-1)!\,\lambda^{-x}e^{\lambda}\left[\wp_\lambda h_{(0)}\wp_\lambda(1-h_{c_0})\right]$$

$$= (x-1)!\,\lambda^{-x}e^{\lambda}\left[e^{-\lambda}(1-e^{-\lambda})\right]$$

$$\leq (x-1)!\,\lambda^{-x}e^{\lambda}\left[e^{-\lambda} - e^{-2\lambda}\right]$$

$$\leq (x-1)!\,\lambda^{-1}(1-e^{-\lambda})$$

$$= (x-1)!\,\lambda^{-1}\left[1 - 1 - \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \cdots \ldots\right]$$

$$= \frac{(x-1)!}{1!}\lambda^1\left[\frac{-\lambda^1}{1!} + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \cdots.\right].$$

If $x_0 = x > 0$, then

$$g_A(1) \leq \frac{\lambda^{-1}}{x_0}(1-e^{-\lambda}).$$

Thus

$$|\Delta g_A(x)| = |g_A(x+1) - g_A(x)| \leq |g_A(1)|\forall x \in \mathbb{N}$$

$$|\Delta g_A(x)| \leq |g_A(1)| \leq \frac{\lambda^{-1}}{x_0}(1-e^{-\lambda})$$

$$|\Delta g_A(x)| \leq \frac{\lambda^{-1}}{x_0}(1-e^{-\lambda}). \qquad (16)$$

Combining (13)and (16) gives

$$|\Delta g_A(x)| \leq \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}\right\}. \qquad (17)$$

2.2. ω – Function

ω- function had been studiedand used by many authors. Teerepabolan [7, 8] and Wongkasem and Teerepabolan [9] among others defined a ω -function associated with non-negative integer valued random variable in the relation

$$\omega(x) = \frac{1}{p(x)\sigma^2}\sum_{k=0}^{x}[\mu - h(x)]p(x), x\epsilon S(X). \qquad (18)$$

Lemma 2.3:let $P_X(k) > 0, \forall\, k \in \mathbb{N}$there exist$w(x)$ such that

$$\frac{P(x-1)}{P(x)} = \frac{x(b+(n-x)N\gamma)}{(n-x+1)(r+(x-1)N\gamma)}. \qquad (19)$$

Lemma 2.4: For $P_X(k) > 0$ there exist

$$\omega(x) = \omega(x-1)\frac{P(x-1)}{P(x)} + \frac{\mu - h(x)}{\sigma^2}\forall\, x \in \mathbb{N}. \qquad (20)$$

Lemma 2.5:Let $\omega(x)$ be the $\omega-$function associated with Pólya random variable $X$and $P_X(k) > 0$, for $0 \leq k \leq n$ then

$$\omega(x) = \frac{(n-x)(r+N\gamma x)}{N\sigma^2}, \qquad (21)$$

where $\sigma^2 = \frac{npq(1+\gamma n)}{(1+\gamma)}$.

Proof

$$\omega(x) = \frac{(n-x)(r+N\gamma x)}{N\sigma^2}$$

and

$$\omega(x-1) = \frac{(n-x+1)(r+(x-1)N\gamma)}{N\sigma^2}. \qquad (22)$$

Substituting (22) in Lemma 2.4 we obtained

$$\omega(x) = \frac{(r+N\gamma x)(n-x)}{N\sigma^2}.$$

Lemma 2.6: If a non-negative valued random variable X have$P_X(k) > 0$, for every x in the support of X, and finite variance $0 < \sigma^2 < \infty$ then

$$E[(X-\mu)g(x)] = \sigma^2 E[\omega(x)\Delta g(x)] < \infty. \qquad (24)$$

And for any function $f: \mathbb{N} \cup \{0\} \to R$ for which$|\omega(x)\Delta g(x)| < \infty$, where $\Delta g(x) = g(x+1) - g(x)$ then $E[\omega(x)] = 1$

3. Main Result

Theorem 3.1: Let X be a Polya random variable with $\lambda = np$ and $(n-1)N\gamma \leq r$ and $c > 0\forall\, x_0 \in \mathbb{N} \cup (0)$ then following holds

1. For $x_0 = 0$

$$d_{tv}(P_X(A), \wp_\lambda(A)) \leq \lambda^{-2}(\lambda + e^{-\lambda} - 1)E|\lambda - \sigma^2\omega(x)| + |\lambda - \mu|E|g_A(X)|$$

2. For $x_0 > 0$

$$d_{tv}\left(P_X(A), \wp_\lambda(A)\right) \leq \min\left\{\lambda^{-1}\left(1-e^{-\lambda}\right), \frac{1}{x_0}\right\} E|\lambda - \sigma^2\omega(x)| + |\lambda - \mu| E|g_A(X)|$$

Proof

Substituting $x$by $X$from the right hand side (9) and taking expectation gives as in [10];

$$E[h_A(x) - \wp_\lambda(h)] = E[\lambda g(X+1) - Xg(X)\,]$$

$$\begin{aligned} d_{tv}\left(P_X(A), \wp_\lambda(A)\right) &= E\lambda[g(X+1)] - E[Xg(X)] \\ &= \lambda E[g(X+1) - g(X) + g(X)] - E[Xg(X)] \\ &= E[\Delta g(X) + g(X)] - E[Xg(X)] \\ &= \lambda E[\Delta g(X) + g(X)] - E[(X-\mu)g(X)] - E[\mu g(X)] \\ &= E[\lambda\Delta g(X) + \lambda g(X)] - E[(\lambda-\mu)g(X)] - E[(X-\mu)g(X)] \quad (24) \end{aligned}$$

From lemma 2.6

$$\begin{aligned} E[(X-\mu)g(X)] &= \sigma^2 E[\omega(X)\Delta g(X)] \\ &= \lambda E[\Delta g(X)] + E[\lambda-\mu]g(X) - \sigma^2 E[\omega(X)\Delta g(X)] \\ &= E\left[\left(\lambda - \sigma^2\omega(X)\right)\Delta g(X)\right] + E[(\lambda-\mu)]g(X) \end{aligned}$$

$$\begin{aligned} |p_X(A) - \wp_\lambda(A)| &= \left| E\left[\left(\lambda - \sigma^2\omega(X)\right)\Delta f(X)\right] + E[(\lambda-\mu)]g(X) \right. \\ &\leq [|\lambda - \sigma^2\omega(X)|\Delta g(X)|] + |\lambda-\mu| E|g(X)| \\ &\leq \sup_{x\geq 1} |\Delta g(X)| E|\lambda - \sigma^2\omega(X)| + |\lambda-\mu| E|g(X)| \end{aligned}$$

By lemma 2.2 and [10] the theorem is proved

Corollary 3.1

If$E|\lambda - \sigma^2\omega(x)| \geq 0, \lambda - \omega(x) = np - \frac{np(1+\gamma n)}{(1+\gamma)}$and $\lambda = \mu$

1. For $x_0 = 0$

$$|P_x(A) - \wp_\lambda(A)| \leq \lambda^{-1}\left(\lambda + e^{-\lambda} - 1\right)\frac{r(N+N\gamma) - b(n-1)N\gamma}{N(N+N\gamma)}. \quad (25)$$

2. For $x_0 > 0$

$$|P_x(A) - \wp_\lambda(A)| \leq \min\left\{\frac{\lambda}{x_0}, \left(1-e^{-\lambda}\right)\right\}\frac{r(N+N\gamma) - b(n-1)N\gamma}{N(N+N\gamma)}. \quad (26)$$

4. Numerical Result

Using the same numerical examples in Egege, et al [1] to illustrate how well Pólya can be approximated by Poisson in terms of total variation distance andits non-uniform upper bound with $c = 1$.

Table 1. Numerical examples of the total distances between Pólya distribution and Poisson distribution and its upper bound.

$x_0, r, n, N, \lambda,$	$P_X(A)$	$\wp_\lambda(A)$	d_{tv}	B_1 $A = \{x_0\} \in \{0\}$	B_2 $A = x_0 \in \mathbb{N}$	Uniformupper
0, 10, 5, 1000, 0.05	0.951085826	0.95122945	0.000143624	0.000148612	$min\left\{\begin{matrix} 0.000294767, \\ \frac{0.05}{x_0}(0.00643956) \end{matrix}\right\}$	0.000294767
1, 10, 5, 1000, 0.05	0.047841339	0.047561471	0.000279868		Min $\left\{\begin{matrix} 0.000294767, \\ 0.000321978 \end{matrix}\right\}$	0.000294767

$x_0, r, n, N, \lambda,$	$P_X(A)$	$\wp_\lambda(A)$	d_{tv}	B_1 $A = \{x_0\} \in \{0\}$	B_2 $A = x_0 \in \mathbb{N}$	Uniformupper
0, 10, 10, 1000, 0.1	0.904790652	0.904837418	0.000046766	0.000053123	$\min\left\{\begin{matrix} 0.0000104574, \\ \frac{0.1}{x_0}(0.0001098901) \end{matrix}\right\}$	0.000104574
2, 10, 10, 1000, 0.1	0.004492181	0.004524187	0.000397623		$\min\left\{\begin{matrix} 0.000104574, \\ 0.000054945 \end{matrix}\right\}$	0.000104574
0, 25, 20, 1000, 0.5	0.605592734	0.606530660	0.000937926	0.001378258	$\min\left\{\begin{matrix} 0.002545169, \\ \frac{0.5}{x_0}(0.006493506) \end{matrix}\right\}$	0.002545169
1, 25, 20, 1000, 0.5	0.304624112	0.303265330	0.001358782		$\min\left\{\begin{matrix} 0.002545169, \\ 0.0016233776 \end{matrix}\right\}$	0.002545169
2, 25, 20, 1000, 0.5	0.075772564	0.075816332	0.000043768		$\min\left\{\begin{matrix} 0.002545169, \\ 0.0016233776 \end{matrix}\right\}$	0.002545169
0, 25, 25, 1000, 0.625	0.535059233	0.535261429	0.000202196	0.0004162264	$\min\left\{\begin{matrix} 0.000754446, \\ \frac{0.626}{x_0}(0.001623377) \end{matrix}\right\}$	0.000754446

Where$B_1 = \lambda^{-1}\left(\lambda + e^{-\lambda} - 1\right)\frac{r(N+N\gamma)-b(n-1)N\gamma}{N(N+N\gamma)}$for $x_0 = 0$

$B_2 = \min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0}\right\}\frac{r(N+N\gamma)-b(n-1)N\gamma}{N(N+N\gamma)}forx_0 \in \mathbb{N}$

Corollary 4.1

1. Min $\left\{\left(1 - e^{-\lambda}\right), \frac{\lambda}{x_0}\right\} < \lambda^{-1}\left(\lambda + e^{-\lambda} - 1\right) < \left(1 - e^{-\lambda}\right)$ for $x_0 = 0$
2. $\lambda^{-2}\left(\lambda + e^{-\lambda} - 1\right) < \min\left\{\lambda^{-1}\left(1 - e^{-\lambda}\right), \frac{1}{x_0}\right\} \leq \lambda^{-1}\left(1 - e^{-\lambda}\right)$for $x_0 > 0$.

5. Discussion

In view of table 1, when n is close to r and very small a good approximation is obtained. The non-uniform upper bound used in this work is a criterion use for measuring the accuracy of approximation of Pólya by Poisson. It was also observed that the result obtained for non-uniform bound is better than uniform bound in [1], which is in agreement with corollary 4.1. If the upper bound is very small then a good approximation is obtained. And the total variation distancewill be close the upper bounds. This is in agreement with [1]. To obtain a small upper bounds and a total distance variation close to the bounds the following must exist

1. $n \leq r$
2. $\frac{r}{N}$ small and$N\gamma$ =1
3. NSufficiently large.

6. Conclusion

In this work, a wider approximation called total variation distances between Poisson and Pólya distribution was considered. Stein's Chen method and $\omega -$ function were applied to give a non-uniform bound. It was found that non-uniform upper bound approximation of Pólya distribution by Poisson distribution is better than uniform upper bound provided r is close to $n$and$N$ is large.

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