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Invariant Differential Operator on Homogeneous Space

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Abstract

This paper is the study of invariant differential operator on homogeneous space. We study how a manifold structure for the quotient group of a Lie group can be seen as a homogeneous space and the invariant differential operator on homogeneous space is commutative when homogeneous space symmetric.

Keywords

Lie Group, Lie Algebra, Isotropy Group, Manifold, PBW Theorem and Symmetric Space

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1. Introduction

In the area of mathematics, most importantly in the theory of Lie group a homogeneous space for a group is a non empty manifold or topological space on which the group acts continuously by symmetry in a transitive way.

In the theory of Lie algebra, the Poincaré-Birkhoff-Witt (Poincaré (1990), G.D. Birkhoff (1937), Witt (1937); frequently contracted to PBW theorem) is a result giving an explicit description of the universal enveloping algebra of a Lie algebra. The 'PBW type theorem' or even 'PBW theorem' may also refer to various analogues of the original theorem, comparing a filtered algebra to its associated graded algebra, in particular, in the area of quantum groups.

In differential geometry, representation theory and harmonic analysis, a symmetric space is a smooth manifold whose group of symmetry contains inversion symmetry of every point. Riemannian symmetric spaces arise in a wide variety of situation in both Mathematics and Physics.

2. Enveloping Algebra [3]

In this section, we discuss the definition of enveloping algebra and some related topics of enveloping algebra. The universal enveloping algebra is an alternative (algebraic) version of the Campbell-Baker-Hausdorff theorem. Also, it depends on several notions which are extremely important in their own right, so we pause to develop them.

A map $\gamma: \mathfrak{g} \to U\mathfrak{g}$ is called a universal algebra of a Lie algebra $\mathfrak{g}$, where $U\mathfrak{g}$ is an associative algebra with unit, if following conditions are satisfied:

a. γ is a Lie algebra homomorphism, i.e

it is linear and $\gamma[x, y] = \gamma(x)\gamma(y) - \gamma(y)\gamma(x)$.

b. If H is any associative algebra with unit and $\beta: \mathfrak{g} \to H$ is any Lie algebra homomorphism then there exists a unique homomorphism h of associative algebras such that $\beta = h \circ \gamma$.

It is also noted that if $U\mathfrak{g}$ exists, it is unique up to a unique isomorphism. Then, such $U\mathfrak{g}$ is called universal enveloping

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algebra. Now if $\mathfrak{g}$ is the Lie algebra of left invariant vector fields on a group G, then $\mathfrak{g}$ can be considered as a Lie algebra consisting of all left invariant first order homogeneous differential operators on G. Then $U\mathfrak{g}$ to consist of all left invariant differential operators on G.

Now we want to construct $U\mathfrak{g}$. Let, D be the ring of all left differential operators on G and D^n consisting of those differential operators with total degree at most n. The quotient D^n/D^{n-1} consists of those homogeneous differential operators of degree n. When we consider the left invariant differential operators on a group, then we can take vector fields to be left invariant and all the function coefficients to be constant. Thus $(U\mathfrak{g})^n/(U\mathfrak{g})^{n-1}$ consists of all symmetric polynomial expressions, homogeneous of degree n. This is the content of the Poincaré-Birkhoff-Witt theorem.

Now, Let, $V = \mathfrak{g}$ be a Lie algebra and let J be the two sided ideal in $T\mathfrak{g}$ generated the elements $[p, q] - p \otimes q + q \otimes p$ then, $U\mathfrak{g} = T\mathfrak{g}/J$ is a universal algebra for $\mathfrak{g}$.

Again if β is any homomorphism from $\mathfrak{g}$ into an associative algebra H , then β extends to a unique algebra homomorphism $\mu: T\mathfrak{g} \longrightarrow H$ which must vanish on J only if it is to be a Lie algebra homomorphism.

If $d: \mathfrak{g} \longrightarrow R$ is a Lie algebra homomorphism, then the composition $\sigma_R \circ d: \mathfrak{g} \longrightarrow UR$ induces a homomorphism $U\mathfrak{g} \longrightarrow UR$ and this assignment sending the Lie algebra homomorphism into associative algebra homomorphism, which is functorial.

Here, we consider $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \sigma_i: \mathfrak{g}_i \longrightarrow U(\mathfrak{g}_i)$ and $\sigma: \mathfrak{g} \to U\mathfrak{g}$ are the canonical homomorphism.

Let us define,

$$f: \mathfrak{g} \to U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2),$$

$$f(x_1 + x_2) = \sigma_1(x_1) \otimes 1 + 1 \otimes \sigma_2(x_2).$$

This is a homomorphism because both the x_1 and x_2 commute. Thus we get a automorphism

$$\varphi: U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2).$$

Also, $x_1 \longmapsto \sigma(x_1)$ is a Lie algebra homomorphism of $\mathfrak{g}_1 \to U(\mathfrak{g})$ which thus extends to a unique algebra homomorphism $\alpha_1: U(\mathfrak{g}_1) \to U(\mathfrak{g})$ and similarly, $\alpha_2: U(\mathfrak{g}_2) \to U(\mathfrak{g})$.

We have, $\alpha_1(x_1)\alpha_2(x_2) = \alpha_2(x_2)\alpha_1(x_1), x_1 \in \mathfrak{g}_1, x_2 \in \mathfrak{g}_2$, since $[x_1, x_2] = 0$.

As the $\sigma_i(x_i)$ generate $U(\mathfrak{g}_i)$ the above equation holds with x_i replaced by arbitrary element, $u_i \in U(\mathfrak{g}_i), i = 1,2$.

So we have a homomorphism

$$\alpha: U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \longrightarrow U(\mathfrak{g});$$

That is, $\alpha(u_1 \otimes u_2) = \alpha_1(u_1)\alpha_2(u_2)$.

Then, we have, $\alpha \circ \varphi(x_1 + x_2) = \alpha(x_1 \otimes 1) + \alpha(1 \otimes x_2) = x_1 + x_2$.

So $\alpha \circ \varphi = id$ on $\mathfrak{g}$ and hence on $U(\mathfrak{g})$ and $\varphi \circ \alpha(x_1 \otimes 1 + 1 \otimes x_2) = x_1 \otimes 1 + 1 \otimes x_2$.

So $\varphi \circ \alpha = id$ on $\mathfrak{g}_1 \otimes 1 + 1 \otimes \mathfrak{g}_2$ and hence on $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$.

Thus, $U(\mathfrak{g}_1 \otimes \mathfrak{g}_2) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$.

3. Criterion for a Differential Operator to Be Invariant [7]

Let, M be a differentiable manifold of dimension n and $C^\infty(M)$ denotes the space of complex functions on M. Again, let, $k \in C^\infty(M)$ and (φ, V) is a local chart on M. We suppose that $k \circ \varphi^{-1}$ is a composite function defined on $\varphi(V)$ and we denote it by $k^* = k \circ \varphi^{-1}$. Also, Let, $z_1, z_2, \dots \dots, z_n$ be the coordinate functions of φ and $\beta = (\beta_1, \beta_2, \dots \dots, \beta_n)$ be a n-tuple of non-negative integers. We also consider

$$\partial_i = \frac{\partial}{\partial x_i} \text{ for } i = 1,2, \dots \dots, n \text{ and}$$

$$D^\beta = \partial_1^{\beta_1}, \partial_2^{\beta_2}, \dots \dots, \partial_n^{\beta_n}.$$

Let, $C_c^\infty(M)$ be the set of all functions with compact support of $C^\infty(M)$.

Now, a linear map $D: C_c^\infty(M) \to C_c^\infty(M)$ is called a differential operator on M if the following condition is satisfied:

For each $q \in M$ and each local chart (φ, V) around q there exists a finite set of functions $b_\beta \in C^\infty(V)$ such that for each $k \in C_c^\infty(M)$ is contained in V and

$$(Dk)(y) = \textstyle\sum_\beta b_\beta(y)\left[D^\beta k^*\right](\varphi(y)) \text{ if } y \in V, (Dk)(y) = 0 \text{ if } y \notin V \qquad (1)$$

Now, let, L be a Lie group, then L is said to be a Lie transformation group of M if for each $l \in L$ which is associated with a diffeomorphism $\sigma(l)$ of M such that

(a) $\sigma(l_1 l_2) = \sigma(l_1)\sigma(l_2)$ for all $l_1, l_2 \in L$.

(b) $(l, m) \longmapsto \sigma(l)m$ is a differentiable mapping from $L \times M$ onto M.

If the action is transitive, then M is called a homogeneous space. In this case we can say that M is diffeomorphic to the quotient manifold L/N of left cosets lN, where N denotes the isotropy group of some element in M. The action on L/N is given left multiplication $\sigma(l)(xN) = lxN$ for all $l, x \in L$.

Now suppose that L is a transitive Lie transformation group of M. A differential operator D on M is called L-invariant if for all $k \in C^{\infty}(M)$ and all $l \in L$,

$$D(\sigma(l)k) = \sigma(l)(Dk) \quad (2)$$

where, $[\sigma(l)k](q) = k(\sigma(l)q)$ for all $q \in M$.

Now, we consider the notation $\mathcal{D}(m)$ for the algebra of L-invariant differential operators on M. For each $D \in \mathcal{D}(m)$, we fix $q \in M$ and we consider a local chart (φ, V) around q. Then D has a local expression near q which is given in equation (1). We define a polynomial in r variables $R_1, R_2 \ldots \ldots, R_r$ by $P(R_1, R_2 \ldots, R_r) = \sum_\beta b_\beta\,(q) R_1^{\beta_1} R_2^{\beta_2} \ldots \ldots R_r^{\beta_r}$.

$$\text{Now, } Dk(q) = [P(\partial_1, \partial_2, \ldots \ldots, \partial_r)k^*](\varphi(q)) \text{ for every } k \in C^{\infty}(M). \quad (3)$$

From (2), we find,

$$Dk(\sigma(l)q) = [P(\partial_1, \partial_2, \ldots \ldots, \partial_r)\sigma(l)k^*](\varphi(q)) \text{ for every } l \in L \text{ and } k \in C^{\infty}(M). \quad (4)$$

Now we can say that D is uniquely determined by polynomial P because the action of L on M is transitive.

Suppose that W is a linear space of finite dimension over a field K of characteristic 0 and $T(W)$ denote the tensor algebra over W and I be the ideal of $T(W)$ generated by the set of element of the form $M \otimes N - N \otimes M$; $M, N \in W$. Then, the factor algebra $\mathcal{S}(W) = T(W)/I$ is called the symmetric algebra over W. Also, If $Q_1, Q_2 \ldots \ldots, Q_n$ is a basis of W, then $\mathcal{S}(W)$ can be identified with abelian algebra of polynomials in the base element over K.

For a given Lie group L with Lie algebra $\mathfrak{l}$, let, $\mathcal{S}(\mathfrak{l})$ be the symmetric algebra of $\mathfrak{l}$, where L acts on itself by multiplication. Now, from above discussion every $D \in \mathcal{D}(L)$ determines a unique polynomial $P \in \mathcal{S}(\mathfrak{l})$ such that

$$(Dk)(l) = \left[P(\partial_1, \partial_2, \ldots \ldots, \partial_n)k(l \exp(z_1 Q_1 + \cdots + z_n Q_n)\right)(0), \quad (5)$$

for all $l \in L$ and all $k \in C^{\infty}(L)$.

For $P \in \mathcal{S}(\mathfrak{l})$ arbitrary, (5) defines a left invariant differential operator $\gamma(P)$ on L. Also, The map, which is defined by $\gamma: \mathcal{S}(\mathfrak{l}) \to \mathcal{D}(L)$, is a linear isomorphism.

Now, If $T_1, T_2, \ldots \ldots ., T_q \in \mathfrak{l}$, then $\gamma(T_1 T_2 \ldots \ldots . T_q) = (q!)^{-1} \sum_\tau T_{\tau(1)} T_{\tau(2)} \ldots \ldots . T_{\tau(q)}$, the sum is taken over all permutation τ of $\{1,2, \ldots \ldots, q\}$.

4. Commutativity of Differential Operator on Homogeneous Space

Let, G/K be a homogeneous space of a Lie group G and $D(G/K)$ be the set of invariant differential operator on G/K. And $D(G/K)$ is also a subalgebra of all differential operators on M.

Let, $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$ and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_c$, where $U(\mathfrak{g})K$ denote the subalgebra of all elements invariant for the adjoint action of K. Here, the elements of $U(\mathfrak{g})$ act on G as left invariant differential operators is the action which is generated by

$$Mf(G) = \frac{d}{dt}|_{t=0} f(ge^{tM}), \text{ where, } M \in \mathfrak{g} \text{ and } f \in C^{\infty}(G).$$

If we consider functions on G/Kas right K-invariant function on G, then there exists a natural action of the element of $U(\mathfrak{g})K$ on $C^{\infty}(G/K)$. Now we can easily verified that this action is an action of differential operators on G/K and we get a homomorphism of algebras $q: U(\mathfrak{g})K \to D(G/K)$ and $U(\mathfrak{g})K \cap U(\mathfrak{g})\mathfrak{h}$ is an ideal of $U(\mathfrak{g})K$ and it is annihilated by q. So we get a homomorphism q, from the quotient $U(\mathfrak{g})K/(U(\mathfrak{g})K \cap U(\mathfrak{g})\mathfrak{h})$ into $D(G/K)$. Now we have following proposition:

Proposition 1[2]: Assume that an K-invariant complement in $\mathfrak{g}$ belongs to $\mathfrak{h}$. Then q is an isomorphism from the algebra $U(\mathfrak{g})K/(U(\mathfrak{g})K \cap U(\mathfrak{g})\mathfrak{h}_c)$ onto $D(G/K)$.

Now, here we assume that G/K is a semisimple symmetric space then we can apply *proposition 1*, since q is K-invariant. We know that an element of $D(G/K)$ is the Laplacian operator L on G/K. Now, on any pseudo-Riemannian manifold this is defined in local coordinates by

$$L = \frac{1}{\sqrt{|\det m|}} \sum_{i,j} \partial_j \sqrt{|\det m|} m^{ij} \partial_i,$$

where, $m = m_{ij}$ is the pseudo-Riemannian structure and m^{ij} is the inverse matrix. Since pseudo-Riemannian structure is invariant so, L is an invariant differential operator. We also have the Casimir element Ω in $U(\mathfrak{g})$ defined by $\Omega = \sum_{i,j} \gamma^{ij} X_i X_j$, where X_i is a basis of $\mathfrak{g}$ and γ^{ij} is the inverse matrix of $B(X_i, X_j)$. It can be seen that L and $q(\Omega)$ coincide up to a positive scalar multiple. Now we will first give the description of $D(G/N)$, then we also give the explanation of $D(G/K)$, where G/K denotes the semisimple symmetric space.

The decomposition of $D(G/N)$ is based on the Iwasawa decomposition $\mathfrak{g} = b_0 \oplus a_0 \oplus d$ and on the PBW theorem. From these we get, $U(\mathfrak{g}) = (b_{0,c} U(\mathfrak{g}) \oplus U(\mathfrak{g}) d_c) \oplus U(a_0)$. Now we define a map ℓ_0 from $U(\mathfrak{g})$ to $U(a_0)$ which is the projection with respect to this decomposition. Since a_0 is abelian, so we can find its universal enveloping algebra with its symmetric algebra which is denoted by $S(a_0)$ instead of $U(a_0)$. Now, we can easily examine that the restriction of ℓ_0 to $U(\mathfrak{g})N$ is a homomorphism and ℓ_0 annihilates $U(\mathfrak{g})d_c$.

From the above proposition, we can say that ℓ is a homomorphism from $D(G/N)$ to $S(a)$. This homomorphism is called the Harish-Chandra homomorphism. It is also denoted by $'_{\gamma_0}$.

Now, let $d_0 \in a_0^*$ be given by

$$d_0 = \frac{1}{2}\sum_{\alpha \in \Sigma^+(a_0, \mathfrak{g})} t_\alpha \alpha$$

which is the half of the trace of ad on b_0 and let, T_{d_0} be the automorphism of $S(a_0)$ generated by $T_{d_0}(z) = z + d_0(z)$ for $z \in a_0$. We define a map $e_0: U(\mathfrak{g})K \to S(a_0)$ by $e_0 = T_{d_0} \circ \ell_0$, which is called the Harish-Chandra isomorphism because of the following theorem.

Theorem 2[2]: We know, the map e_0 is an algebra isomorphism from $D(G/N)$ to $S(a_0)V_0$ is the set of V_0-invariant element in $S(a_0)$ and it is independent of the choice of $\Sigma^+(a_0, \mathfrak{g})$.

Proof: We want to show that $e_0(D)$ is a V_0-invariant differential operator D and e_0 is bijective. We Consider the spherical function f_α on G/N. Also, Let, $L: G \to a_0$ be the Iwasawa projection.

$$\text{Then, } f_\alpha(G) = \int e^{-(\alpha+\beta_0)L(g^{-1}u)}\, du, \qquad (6)$$

where, $\alpha \in a_{0,c}^*$ and $g \in G$. It is clear that f_α is smooth function on G/N. At this moment we have the following two results:

(a) For all $D(G/N)$, the eigenfunctions are the spherical functions. Thus, we have the $Df_\alpha = e_0(D,\alpha)f_\alpha$, for all $D \in D(G/N)$. The integrand in (6) is already an eigenfunction with this eigenvalue.

(b) So we have $f_{\beta\alpha} = f_\alpha$ for all $\beta \in V_0$.

From (a) and (b) we can say that, $e_0(D,\beta\alpha) = e_0(D,\alpha)$. From these we can say that $D(G/N)$ is commutative. Now, we generalize this result for G/K. From the definition of Caftan subspace, we know that, a Caftan subspace for G/K is a maximal abelian subspace of W, consisting of semisimple elements. If there exists a Caftan subspace a_1 containing a_w. Then, $a_w = a_1 \cap W$. The elements of $ad\ a_1$ will be complex eigenvalues. Now, here we had a root system $\Sigma(a_{1c}, \mathfrak{g}_c)$ for a_0, a_w and corresponding to each choice of positive set $\Sigma^+(a_{1c}, \mathfrak{g}_c)$, an analog of the Iwasawa decomposition$\mathfrak{g}_c = b_1 \oplus a_{1c} \oplus \mathfrak{h}_c$, where b_1 is the sum of the root space corresponding to the positive roots.

Now, for more general result we want to construct the Harish-Chandra homomorphism which can be generalized by setting a map $\ell_0: U(\mathfrak{g}) \to U(a_1)$ which is defined by a projection with respect to the Iwasawa decomposition and this gives a homomorphism from $D(G/K)$ to $S(a_1)$. Now, similar to the before if we define $e = T_{d_1} \circ \ell_0$, where $d_1 \in a_{1c}^*$ is half of the trace of ad on b_1 and if we denote the Weyl group of $\Sigma(a_{1c}, \mathfrak{g}_c)$ by W_1, then we have the following theorem:

Theorem 3[2]: The map ℓ is an algebra isomorphism from $D(G/K)$ to $S(a_1)W_1$. Here, It is also independent of the choice of $\Sigma^+(a_{1c}, \mathfrak{g}_c)$.

Proof: We have already seen that in *Theorem 2* $D(G/K)$ is isomorphic to $U(\mathfrak{g})K/(U(\mathfrak{g})K \cap U(\mathfrak{g})\mathfrak{h})$.

Define, $\mathfrak{g}^n = d \cap \mathfrak{h} \oplus p \cap w \oplus i(d \cap q \oplus w \cap \mathfrak{h}) \subset \mathfrak{g}_c$, then $\mathfrak{g}^n$ is a real semisimple Lie algebra with the same complexification as $\mathfrak{g}$.

Let,

$d^n = d \cap \mathfrak{h} \oplus i(w \cap \mathfrak{h}) = \mathfrak{h}_c \cap \mathfrak{g}^n$, $p^n = p \cap w \oplus i(d \cap w) = w_c \cap \mathfrak{g}^n$.

Then, $\mathfrak{g}^n = d^n \oplus p^n$ is a Cartan decomposition of $\mathfrak{g}^n$ and the pair $(\mathfrak{g}^n, d^n)$ is called the noncompact Riemannian form of the pair $(\mathfrak{g}, \mathfrak{h})$. Let, $a_0^n = a_w \oplus i(a_1 \cap d) = a_{1c} \cap \mathfrak{g}^n$. Now, since a_0^n and a_1 have the same complexification, thus a_0^n is a maximal abelian subspace of p^n. Also, the root system $\Sigma(a_{1c}, \mathfrak{g}_c)$ is essentially the same as the root system $\Sigma(a_0^n, \mathfrak{g}_n)$ and their root spaces in $\mathfrak{g}_c$ are identical. Let, (G^n, N^n) be a symmetric pair with $(\mathfrak{g}^n, d^n)$, then G^n/N^n is a symmetric space. Now by *Theorem 2* we get a Harish-Chandra isomorphism ℓ_0^n on G^n/N^n from $U(\mathfrak{g}^n)N^n/(U(\mathfrak{g}^n)N^n \cap U(\mathfrak{g}^n)d_c^n)$ to $S(a_0^n)W^n$.

Since, we know that $U(\mathfrak{g}^n)N^n = U(\mathfrak{g})\mathfrak{h}_c = U(\mathfrak{g})K$ and $S(a_0^n) = S(a_1)W_1$. Also, Thus, ℓ_0^n is identical with ℓ. So, for $D(G/N)$, $D(G/K)$ is a polynomial algebra with $\dim a_1$ of independent generators and in particular $D(G/K)$ is commutative. In the terminology of the proof above actually we have,

$$D(G/K) \cong D(G^n/N^n).$$

5. Conclusion

Here, we want to show that $D(G/K)$ is not commutative if G/K is not symmetric. So, let, $G = SL(2,\mathbb{R})$ be a Lie group. Then, if we consider $K = \{e\}$ (the trivial group). In this case, G/K is diffeomorphic to $G = SL(2,\mathbb{R})$ is not a symmetric space and $D(G/K)$ is isomorphic to the enveloping algebra of $sl(2,\mathbb{C})$, which is far from being commutative.

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