International Journal of Mathematics and Computational Science
Vol. 2, No. 2, 2016, pp. 55-63
http://www.aiscience.org/journal/ijmcs
ISSN: 2381-7011 (Print); ISSN: 2381-702X (Online)

Transportation Inequalities for Stochastic Neutral Delay Evolution Equations Driven by Sub-fractional Brownian Motion

Liping Xu*

School of Information and Mathematics, Yangtze University, Jingzhou, China

Abstract

In this paper, we discuss stochastic neutral partial differential equations of retarded type driven by sub-fractional Brownian motion with Hurst parameter H>1/2. Using the Girsanov transformation argument we establish the quadratic transportation inequalities for the law of the mild solution of those equations driven by sub-fractional Brownian motion under the d_2 metric and the uniform metric d_∞. Last, one example is given to illustrate the feasibility and effectiveness of results obtained.

Keywords

Transportation Inequalities, Girsanov Transformation, Delay SPDEs, Sub-fractional Brownian Motion

Received: March 20, 2016 / Accepted: April 10, 2016 / Published online: May 13, 2016

1. Introduction

In the past decade, a plenty of results have been published concerning Talagrand-type transportation cost inequalities (TCIs) on the path spaces of stochastic processes, see e.g. [1-3] for diffusion processes on R^d, [4] for multidimensional semi-martingales, [5] for diffusion processes with history-dependent drift, [6-7] for diffusion processes on Riemannian manifolds, [8] for SDEs driven by pure jump processes, and [9] for SDEs driven by both Gaussian and jump noises.

Moreover, many different arguments have been developed to establish the transportation inequality. Among others, the Girsanov transformation argument introduced in [1] has been efficiently applied, see e.g. [3] for infinite-dimensional dynamical systems, [4] for time-inhomogeneous diffusions, [5] for multi-valued SDEs and singular SDEs, [10] for neutral functional SDEs and [11] for SDEs driven by a fractional Brownian motion, [12] for neutral functional stochastic evolution equations driven by fractional Brownian motion. Following this line, in this paper we aim to establish transportation inequalities for the law of the mild solution of a class of stochastic functional differential equations driven by sub-fractional Brownian motion with Hurst parameter S>1/2 in infinite dimensional space, which is unknown so far.

Let us now consider the kinds of inequalities we will deal with. To measure distances between probability measures, we use transportation distance, also called Wasserstein distance. Let (E, d) be a metric space equipped with a σ-field β such that the distance d is $\beta \otimes \beta$-measurable. Given $p \geq 1$ and two probability measure μ and ν on E, the Wasserstein distance is defined by:

$$W_p^d(\mu,\nu) = \inf_{\pi \in \varphi(\mu,\nu)} \left(\int\int d(x,y)^p d\pi(x,y)\right)^{\frac{1}{p}}$$

Where $\phi(\mu,\nu)$ denotes the totality of probability measures on $E \times E$ with the marginal μ and ν.

The relative entropy of ν with respect to μ is defined as

* Corresponding author
E-mail address: xlp211@126.com

$$H(\nu|\mu)=\begin{cases}\int \ln\frac{d\nu}{d\mu}d\nu, \nu<<\mu\\ +\infty, otherwise\end{cases}.$$

The probability measure μ satisfies the L^p transportation inequality on (E,d) if there exists a constant $C\geq 0$ such that for any probability measure ν,

$$W_p^d(\mu,\nu)\leq\sqrt{2CH(\nu|\mu)}.$$

As usual, we write $\mu\in T_p(C)$ for this relation. The properties $T_2(C)$ are of particular interest.

As an extension of Brownian motion, recently, Bojdecki et al. [13] introduced and studied a rather special class of self-similar Gaussian process. This process arises from occupation time fluctuations of branching particle systems with poisson initial condition. This process is called the sub-fractional Brownian motion (sub-fBm in short). The sub-fBm with index H∈(0,1) is mean zero Gaussian $S^H=\{S^H(t),t\geq 0\}$ starting from zero, with covariance

$$C_H(t,s)=E\left[S^H(t)S^H(s)\right]=s^{2H}+t^{2H}-\frac{1}{2}\left[(s+t)^{2H}+|t-s|^{2H}\right]$$

for all $s,t\geq 0$. For H=1/2, S^H coincides with the standard Brownian motion B. S^H is neither a semimartingale nor a Markov process when $H\neq\frac{1}{2}$, so many of the power techniques from stochastic analysis are not available when dealing with S^H. The sub-fBm has properties analogous to those of fBm (self-similarity, long-range dependence, Holder paths), but it do not have stationary increments. More works for sub-fractional Brownian motion can be found in Bojdecki et al. [14-15], Tudor [16-21] and Yan and Shen [22]. In this paper, we desire to investigate the properties $T_2(C)$ for law of the mild solution of stochastic neutral delay evolution equations driven by sub-fractional Brownian motion with Hurst parameter H>1/2 under the L^2 metric and the uniform metric.

The contents of the paper are as follows. In Section 2, we present some necessary preliminaries. In Section 3, we investigate the properties $T_2(C)$ for law of the solution of neutral stochastic delay evolution equations driven by sub-fractional Brownian motion with Hurst parameter H>1/2 under the L^2 metric and the uniform metric. In Section 4, we analyze an example to illustrate the results obtained in the work.

2. Preliminaries

In this section we introduce the sub-fractional Brownian motion as well as the Wiener integral with respect to it. We also establish some important results which will be needed throughout this paper.

Let (Ω,F,P) be a complete probability space. Now we aim at introducing the Wiener integral with respect to one-dimensional sub-fBm S^H. Fix a time interval [0, T]. We denote by ε the linear space of R-valued step functions on [0, T], that is, $\varphi\in\varepsilon$ if

$$\varphi(t)=\sum_{i=1}^{n-1}x_iI_{[t_i,t_{i+1}]}(t),$$

where $t\in[0,T]$, $x_i\in R$ and $0=t_1<t_2<\cdots<t_n=T$. For $\varphi\in\varepsilon$ we define its Wiener integral with respect to S^H as

$$\int_0^T\varphi(s)dS^H(s)=\sum_{i=1}^{n-1}x_i\left(S_{t_{i+1}}^H-S_{t_i}^H\right).$$

Let H_{S^H} be the canonical Hilbert space associated to the sub-fBm S^H. That is, H_{S^H} is the closure of the linear span ε with respect to the scalar product

$$\left\langle I_{[0,t]},I_{[0,s]}\right\rangle_{H_{S^H}}=C_H(t,s).$$

We know that the covariance of sub-fractional Brownian motion can be written as

$$E\left[S^H(t)S^H(s)\right]=\int_0^t\int_0^s\phi_H(u,v)dudv=C_H(t,s) \quad (1)$$

Where $\varphi_H(u,v)=H(2H-1)\left[|u-v|^{2H-2}-(u+v)^{2H-2}\right]$.

Formulae (2.1) implies that

$$\langle\varphi,\psi\rangle_{H_{S^H}}=\int_0^t\int_0^t\varphi_u\psi_v\phi_H(u,v)dudv \quad (2)$$

for any pair step functions φ and ψ on [0,T]. Consider the kernel

$$n_H(t,s)=\frac{2^{1-H}}{\Gamma(H-\frac{1}{2})}s^{\frac{3}{2}-H}\left(\int_s^t(x^2-s^2)^{H-\frac{3}{2}}dx\right)I_{[0,t]}(s). \quad (3)$$

By Dzhaparidze and Van Zanten [23], we have

$$C_H(t,s)=c_H^2\int_0^{s\wedge t}n_H(t,u)n_H(s,u)du \quad (4)$$

Where $c_H^2=\frac{\Gamma(1+2H)\sin(\pi H)}{\pi}$. Property (4) implies that $C_H(t,s)$ is non-negative definite. Consider the linear operator

n_H^* from ε to $L^2([0,T])$ defined by

$$n_H^*(\varphi)(s) := c_H \int_s^r \varphi_r \frac{\partial_{n_H}}{\partial r}(r,s)dr \ .$$

Using (2) and (4) we have

$$\begin{aligned}\left\langle n_H^*\phi, n_H^*\psi\right\rangle_{L^2([0,T])} &= c_H^2 \int_0^T \left(\int_s^T \phi_r \frac{\partial n_H}{\partial r}(r,s)dr\right)\left(\int_s^T \psi_u \frac{\partial n_H}{\partial u}(u,s)du\right)ds \\ &= c_H^2 \int_0^T \int_0^T \left(\int_0^{r\wedge u} \frac{\partial n_H}{\partial r}(r,s)\frac{\partial n_H}{\partial u}(u,s)ds\right)\phi_r \psi_u drdu \\ &= c_H^2 \int_0^T \int_0^T \frac{\partial^2 n_H}{\partial r \partial u}(r,u)\phi_r \psi_u drdu \\ &= H(2H-1)\int_0^T \int_0^T [|u-r|^{2H-2} - (u+r)^{2H-2}]\phi_r \psi_u drdu \\ &= \langle \varphi, \psi \rangle_{\mathrm{H}_{S^H}} \ . \end{aligned} \qquad (5)$$

As a consequence, the operator n_H^* provides an isometry between the Hilbert space $\langle \varphi,\psi\rangle_{\mathrm{H}_{S^H}}$ and $L^2([0,T])$. Hence, the process W defined by

$$W(t) := S^H((n_H^*)^{-1}(I_{[0,t]}))$$

is a Wiener process, and S^H has the following Wiener integral representation:

$$S^H(t) = c_H \int_0^t n_H(t,s)dW(s)$$

because $(n_H^*)(I_{[0,t]})(s) = c_H n_H(t,s)$. By Dzhaparidze and Van Zanten[23], we have

$$W(t) = \int_0^t \psi_H(t,s)dS^H(s)$$

where

$$\psi_H(t,s) = \frac{S^{H-1/2}}{\Gamma(3/2-H)} \times \\ \times\left[t^{H-3/2}(t^2-s^2)^{1/2-H} - (H-3/2)\int_s^t (x^2-s^2)^{1/2-H} x^{H-3/2}dx\right] \times \\ \times I_{[0,t]}(s)$$

In addition, for any $\varphi \in \mathrm{H}_{S^H}$,

$$\int_0^T \varphi(s)dS^H(s) = \int_0^T (n_H^*\varphi)(t)dW(t)$$

if and only if $n_H^*\varphi \in L^2([0,T])$.

Also denoting $L^2_{\mathrm{H}_{S^H}}([0,T]) = \{\varphi \in \mathrm{H}_{S^H}, n_H^*\varphi \in L^2([0,T])\}$. Since H>1/2, we have by (5) and Lemma 2.1 of [24]

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset L^2_{\mathrm{H}_{S^H}}([0,T]) \cdot \qquad (6)$$

Moreover, the following useful result holds:

Lemma 2.1 (Nualart [25]) For $\varphi \in L^{\frac{1}{H}}([0,T])$

$$H(2H-1)\int_0^T \int_0^T |\varphi_r||\varphi_u||u-r|^{2H-2}drdu \le C_H \|\varphi\|_{L^{\frac{1}{H}}([0,T])}$$

where $C_H = \left(\dfrac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}\right)^{1/2}$.

Next we are interested in considering a sub-fBm with values in Hilbert space and giving the definition of the corresponding stochastic integral.

Let $(U, \|\cdot\|_U, \langle\cdot\rangle_U)$ and $(K, \|\cdot\|_K, \langle\cdot\rangle_K)$ be two separable Hilbert spaces. Let $L(K,U)$ denote the space of all bounded linear operators from K to U. Let $Q \in L(K,K)$ be a non-negative self-adjoint operator. Denote by $L_Q^0(K,U)$ the space of all $\xi \in L(K,U)$ such that $\xi Q^{1/2}$ is a Hilbert-Schmidt operator. The norm is given by

$$\|\xi\|^2_{L_Q^0(K,U)} = \|\xi Q^{1/2}\|^2_{HS} = tr(\xi Q \xi^*) \ .$$

Then ξ is called a Q-Hilbert-Schmidt operator from K to U.

Let $\{S_n^H(t)\}_{n\in N}$ be a sequence of one-dimensional standard sub-fractional Brownian motions mutually independent on (Ω, F, P). When one considers the following series

$$\sum_{n=1}^\infty S_n^H(t)Q^{1/2}e_n, \quad t \ge 0$$

where $\{e_n\}_{n\in N}$ is a complete orthonormal basis in K, this series does not necessarily converge in the space K. Thus we consider a K-valued stochastic process $S_Q^H(t)$ given formally by the following series:

$$S_Q^H(t) = \sum_{n=1}^\infty S_n^H Q^{1/2}(t)e_n, \quad t \ge 0 \ .$$

If Q is a non-negative self-adjoint trace class operator, then this series converges in the space K, that is, it holds that $S_Q^H(t) \in L^2(\Omega, K)$. Then, we say that the above $S_Q^H(t)$ is a K

-valued Q-cylindrical sub-fractional Brownian motion with covariance operator Q. For example, if $\{\sigma_n\}_{n\in N}$ is a bounded sequence of non-negative real numbers such that $Qe_n=\sigma_n e_n$, assuming that Q is a nuclear operator in K (that is, $\sum_{n=1}^{\infty}\sigma_n<\infty$), then the stochastic process

$$S_Q^H(t)=\sum_{n=1}^{\infty}S_n^H(t)Q^{1/2}e_n=\sum_{n=1}^{\infty}\sqrt{\sigma_n}S_n^H(t)e_n,\quad t\ge 0,$$

is well-defined as a K-valued Q-cylindrical sub-fractional Brownian motion.

Let $\varphi:[0,T]\to L_Q^0(K,U)$ such that

$$\sum_{n=1}^{\infty}\left\|n_H^*(\varphi Q^{1/2}e_n)\right\|_{L^2([0,T];U)}<\infty. \tag{7}$$

Let $\varphi:[0,T]\to L_Q^0(K,U)$ satisfy (2.7). Then, its stochastic integral with respect to the sub-fBm S_Q^H is defined, for $t\ge 0$, as follows

$$\int_0^t\phi(s)dS_Q^H(s):=\sum_{n=1}^{\infty}\phi(s)Q^{1/2}e_n dS_n(s)=\sum_{n=1}^{\infty}(n_H^*(\phi Q^{1/2}e_n))(s)dW(s).$$

Now we turn to state some notations and basic facts about the theory semi-groups and fractional power operators. Let $A:D(A)\to U$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t\ge 0}$, of bounded linear operators on U. For the theory of strongly continuous semigroup, we refer to Pazy [26]. We will point out here some notations and properties that will be used in this work. It is well known that there exist $M\ge 1$ and $\lambda\in R$ such that $\|S(t)\|\le Me^{\lambda t}$ t} for every $t\ge 0$. If $(S(t))_{t\ge 0}$ is a uniformly bounded and analytic semigroup such that $0\in\rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^{\alpha}$ for $0<\alpha\le 1$, as a closed linear operator on its domain $D(-A)^{\alpha}$. Furthermore, the subspace $D(-A)^{\alpha}$ is dense in U, and the expression

$$\|h\|_{\alpha}=\left\|(-A)^{\alpha}h\right\|_U$$

defines a norm in $D(-A)^{\alpha}$. If U_{α} represents the space $D(-A)^{\alpha}$ endowed with the norm $\|\cdot\|_{\alpha}$, then the following properties are well known (cf. Pazy [26], Theorem 6.13 p.74).

Lemma 2.2 Suppose that the preceding conditions are satisfied.

(1) Let $0<\alpha\le 1$. Then U_{α} is a Banach space.

(2) If $0<\beta\le\alpha$ then the injection $U_{\alpha}\to U_{\beta}$ is continuous.

(3) For every $0<\beta\le 1$ there exists $M_{\beta}>0$ such that

$$\left\|(-A)^{\beta}S(t)\right\|\le M_{\beta}t^{-\beta}e^{-\lambda t},\quad t>0,\quad \lambda>0.$$

3. TCIs for Functional SPDEs Driven by Sub-fBm

Let us consider two fixed real numbers $r>0$ and T>0. We denote by $C=C([-r,0];U)$ the Banach space of all continuous functions from $[-r,0]$ into $L^2(\Omega,U)$ equipped with the sup-norm $\|f\|_C:=\sup\left\{\|f(s)\|_{L^2(\Omega,U)}:f\in C,-r\le s\le 0\right\}$. If $x\in C(-r,T;L^2(\Omega,U))$ for each $t\in[0,T]$ we denote by $x_t\in C(-r,T;L^2(\Omega,U))$ the function defined by $x_t(\theta)=x(t+\theta)$, for $\theta\in[-r,0]$.

We discuss the TCIs for the law of the solution of a class of the following stochastic evolution equations with delay driven by sub-fractional Brownian motion with Hurst parameter H>1/2 on R^n:

$$\begin{cases} d[X(t)+G(t,X_t)]=AX(t)dt+f(t,X_t)dt+\sigma(t)dS_Q^H(t),\\ \qquad t\in[0,T]\\ X_0(t)=\phi(t),t\in[-r,0]\end{cases} \tag{8}$$

where the initial data $\varphi\in C(-r,0;L^2(\Omega,U))$, and $A:D(A)\to U$ is the infinitesimal generator of a strongly continuous semigroup $S(\cdot)$ on U. For Eq. (3.1) we assume that the following conditions hold.

(H1) A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t\ge 0}$, of bounded linear operators on U. Further, to avoid unnecessary notations, we suppose that $0\in\rho(A)$, and that, see Lemma 2.1,

$$\|S(t)\|\le M \text{ and } \left\|(-A)^{(1-\alpha)}S(t)\right\|\le M_{1-\alpha}t^{\alpha-1}$$

for some constants M, $M_{1-\alpha}$ and every $t\in[0,T]$.

(H2) $f:[0,T]\times C([-r,0];U)\to U$ satisfies the following Lipschitz condition: that is, there exists a constant $C_f>0$ such that for any $x,y\in C([-r,0];U)$ and $t\in[0,T]$,

$$\|f(t,x)-f(t,y)\|_U^2 \le C_f \|x-y\|_C^2$$

and

$$\|f(t,x)\|_U^2 \le C_f \left(1+\|x\|_C^2\right).$$

(H3) There exist constants $0<\alpha\le 1$, $C_G \ge 0$ such that the function G is U_α -valued and satisfies for any $x, y \in C([-r,0];U)$,

$$\left\|(-A)^\alpha G(t,x)-(-A)^\alpha G(t,y)\right\| \le C_G \|x-y\|_C,$$

$$G(t,0)\equiv 0 \text{ for } t\ge 0 \text{ and } C_G\left(\left\|(-A)^\alpha\right\|+T^\alpha \frac{M_{1-\alpha}}{\alpha}\right)<1.$$

(H4) The function $(-A)^\alpha G$ is continuous in quadratic mean sense:

for all $\varphi \in C(-r,0;L^2(\Omega,U))$,

$$\lim_{t\to s} E\left\|(-A)^\alpha G(t,x_t)-(-A)^\alpha G(s,y_s)\right\|^2 = 0.$$

(H5) The function $\sigma:[0,T]\to L_Q^0(K,U)$ satisfies

$$\int_0^T \|\sigma(s)\|_{L_Q^0}^2 ds < \infty, \quad \forall T>0 \cdot$$

For the equation (4.1), by operating a similar scheme as in the proofs of Theorem 3.1 in [27], we can obtain the following lemma.

Lemma 3.1 Under the assumptions (H1)-(H5), for every $\varphi \in C(-r,0;L^2(\Omega,U))$, Eq. (8) admits a unique mild solution $\{X(t,\varphi)\}_{t\in[0,T]}$. That is, for any $\varphi \in C(-r,0;L^2(\Omega,U))$ there exists a unique U -valued adapted process $\{X(t,\varphi)\}_{t\in[0,T]}$, which is continuous in $L^2(\Omega,P)$, such that

$$X(t)=S(t)(\phi(0)+G(0,\phi))-G(t,x_t)-\int_0^t AS(t-s)G(s,X_s)ds$$
$$+\int_0^t S(t-s)f(s,X_s)ds+\int_0^t S(t-s)\sigma(s)dS_Q^H(s). \quad (9)$$

Theorem 3.1 Let the conditions (H1)-(H5) hold and and P_φ be the law of $X(t,\varphi)$, solution process of Eq. (8). Assume further that σ is bounded by $\tilde{\sigma}:=\sup_{0\le t\le T}\|\sigma(t)\|_U$. Then the probability measure P_φ satisfies $T_2(C)$ on the metric space $C([0,T];U)$ with

$$C=\frac{2\Gamma(2H-1)}{\Gamma(H-\frac{1}{2})\Gamma(H+\frac{1}{2})}M^2e^{2\rho T}\tilde{\sigma}^2 e^{2T^2M^2e^{2\rho T}C_f}$$

when using the metric

$$d_\infty(r_1,r_2):=\sup_{0\le t\le T}\|r_1-r_2\|_U, \quad r_1,r_2\in C([0,T];U).$$

Proof: It should be pointed out that the argument is motivated by [1], where the key point is to express the finiteness of the entropy by means of the energy of the drift from the Girsanov transformation of a well chosen probability measure.

Let P_φ be the law of $X(t,\varphi)$ on $\Sigma:=C([0,T];U)$ and Q be any probability measure on Σ such that $Q<<P_\varphi$. Define

$$\tilde{Q}:=\frac{dQ}{dP_\varphi}(X(\cdot,\varphi))P, \quad (10)$$

which is a probability measure on (Ω,F). Recalling the definition of entropy and adopting a measure-transformation argument we obtain from (10) that

$$H(\tilde{Q}|P)=\int_\Omega \ln\left(\frac{d\tilde{Q}}{dP}\right)d\tilde{Q}=\int_\Omega \ln\left(\frac{dQ}{dP_\phi}(X(\cdot,\phi))\right)\frac{dQ}{dP_\phi}(X(\cdot,\phi))dP$$
$$=\int_\Sigma \ln\left(\frac{dQ}{dP_\phi}\right)\frac{dQ}{dP_\phi}(X(\cdot,\phi))dP_\phi$$
$$=H(Q|P_\phi).$$

Following [28], there exists a predictable process $h(t)_{0\le t\le T}\in U$ with $\int_0^T \|h(s)\|_U^2 ds<\infty$, P -a.s., such that

$$H(\tilde{Q}|P)=H(Q|P_\varphi)=\frac{1}{2}E^{\tilde{Q}}\|h(t)\|_U^2 dt.$$

Due to the Girsanov theorem, the process $\left(\tilde{W}(t)\right)_{t\in[0,T]}$ defined by

$$\tilde{W}(t)=W(t)-\int_0^t h(s)ds$$

is a Brownian motion with respect to $\{F_t\}_{t\ge 0}$ on the probability space $(\Omega,F,\tilde{Q})$, and is associated (thanks to the transfer principle) with the $\tilde{Q}$-fractional Brownian motion $\left(\tilde{S}_Q^H\right)_{t\in[0,T]}$ defined by

$$\tilde{S}_Q^H = c_H \int_0^t n_H(t,s)d\tilde{W}(s) = c_H \int_0^t n_H(t,s)dW(s) - c_H(n_H h)(t)$$

where the operator defined by $(n_H h)(t) := c_H \int_0^t n_H(t,s)h(s)d(s)$.

Consequently, under the measure $\tilde{Q}$, the process $\{X(t,\varphi)\}_{t\in[0,T]}$ satisfies

$$\begin{cases} d[X(t)+G(t,X_t)] = AX(t)dt + f(t,X_t)dt + \sigma(t)d(n_H h)(t) \\ \qquad + \sigma(t)d\tilde{S}_Q^H(t), t\in[0,T] \\ X_0(t) = \phi(t), t\in[-r,0]. \end{cases} \tag{11}$$

We now consider the solution Y (under $\tilde{Q}$) of the following equation:

$$\begin{cases} d[Y(t)+G(t,Y_t)] = AY(t)dt + f(t,Y_t)dt \\ \qquad + \sigma(t)d\tilde{S}_Q^H(t), t\in[0,T] \\ Y_0(t) = \phi(t), t\in[-r,0]. \end{cases} \tag{12}$$

By the Lemma 3.1, under $\tilde{Q}$ the law of $Y(\cdot)$ is P_φ. Thus (X,Y) under $\tilde{Q}$ is a coupling of $(\tilde{Q},P_\varphi)$, and it follows that

$$[W_2^{d_\infty}(Q,P_\phi)]^2 \le E_{\tilde{Q}}(|d_\infty(X,Y)|^2) = E_{\tilde{Q}}(\sup_{0\le t\le T}\|X(t)-Y(t)\|_U^2).$$

We now estimate the distance between X and Y with respect to d_∞.

Note from (9), (11) and (12) that

$$X(t)-Y(t) = G(t,Y_t)-G(t,X_t) - \int_0^t AS(t-s)[G(s,Y_s)-G(s,X_s)]ds$$

$$+\int_0^t S(t-s)[f(s,X_s)-f(s,Y_s)]ds + c_H\int_0^t S(t-s)\sigma(s)d(n_H h)(s). \tag{13}$$

Recalling a fundamental inequality: for any $a,b>0$ and $\varepsilon\in(0,1)$

$$(a+b)^2 \le a^2/\varepsilon + b^2/(1-\varepsilon)$$

and invoking (13), we have

$$\|X(t)-Y(t)\|_U^2$$

$$\le \frac{1}{\varepsilon}\left\{\|G(t,Y_t)-G(t,X_t)\|_U + \left\|\int_0^t AS(t-s)[G(s,Y_s)-G(s,X_s)]ds\right\|_U\right\}^2 \tag{14}$$

$$+\frac{2}{1-\varepsilon} + \left\|\int_0^t S(t-s)[f(s,X_s)-f(s,Y_s)]ds\right\|_U^2$$

$$+\frac{2}{1-\varepsilon}c_H^2\left\|\int_0^t S(t-s)\sigma(s)d(n_H h)(s)\right\|_U^2$$

$$=: I_1 + I_2 + I_3$$

where we also use the basic inequality $(a+b)^2 \le 2a^2+2b^2$ for any $a,b\in R$. By [26], we obtain that $(-A)^{-\alpha}$ is bounded and

$$(-A)^{\alpha+\beta}x = (-A)^\alpha\cdot(-A)^\beta x$$

For $x\in D((-A)^{-\gamma})$, the domain of $(-A)^{-\gamma}$, with $\gamma=\max\{\alpha,\beta,\alpha+\beta\}$, $\alpha,\beta\in R$. By (H1) and (H3), it follows that

$$I_1(t) = \frac{1}{\varepsilon}(\|(-A)^{-\alpha}(-A)^\alpha G(t,Y_t)-(-A)^{-\alpha}(-A)^\alpha G(t,X_t)\|_U$$

$$+\left\|\int_0^t(-A)S(t-s)(-A)^{-\alpha}[(-A)^\alpha G(s,Y_s)-(-A)^\alpha G(s,Xs)ds\right\|_U)^2$$

$$= \frac{1}{\varepsilon}(\|(-A)^{-\alpha}(-A)^\alpha G(t,Y_t)-(-A)^{-\alpha}(-A)^\alpha G(t,X_t)\|_U$$

$$+\left\|\int_0^t(-A)^{1-\alpha}S(t-s)[(-A)^\alpha G(s,Y_s)-(-A)^\alpha G(s,Xs)ds\right\|_U)^2$$

$$\le\frac{1}{\varepsilon}\left\{C_G\left(\|(-A)^{-\alpha}\|+\int_0^T\|A^{1-\alpha}S(t)\|dt\right)\|X(t)-Y(t)\|_U\right\}^2$$

$$\le\frac{1}{\varepsilon}\left\{C_G\left(\|(-A)^{-\alpha}\|+\frac{T^\alpha M_{1-\alpha}}{\alpha}\right)\|X(t)-Y(t)\|_U\right\}^2$$

where $X(t)=Y(t)$, $t\in[-r,0]$.

Taking $\varepsilon = C_G\left(\|(-A)^{-\alpha}\|+\frac{T^\alpha M_{1-\alpha}}{\alpha}\right)$ by (H3) we obtain that $I_1\le\varepsilon\|X(t)-Y(t)\|_U^2$. Thus, by (14) we get $E\|X(t)-Y(t)\|_U^2 \le \frac{1}{1-\varepsilon}E(I_2(t)) + \frac{1}{1-\varepsilon}E(I_3(t))$. Next, by virtue of Holder's inequality and (H2), we have

$$E(I_2(t)) \le \frac{2TM^2C_f{}^2}{1-\varepsilon}\int_0^t E\|X(s)-Y(s)\|_U^2 ds, \tag{15}$$

here, we use $X=Y$ over the interval $[-r,0]$.

Since $h\in L^2([0,T];U)$, by the view of Holder's inequality, we can obtain

$$\left\|\int_0^t n_H(t,s)h(s)ds\right\|^2 \le \int_0^t \|h(s)\|_U^2 ds \cdot \int_0^t |n_H(t,s)|^2 ds .$$

Next, we estimate $\int_0^t n_H{}^2(t,s)ds$. By the definition of $n_H(t,s)$, we have

$$\int_0^t n_H{}^2(t,s)ds =$$

$$\int_0^t \frac{2^{2-2H}\pi}{\Gamma^2(H-1/2)} s^{3-2H}\left(\int_s^t (x-s)^{H-3/2}(x+s)^{H-3/2}dx\right)^2 ds$$

$$\le \frac{2^{2-2H}\pi}{\Gamma^2(H-1/2)}\int_0^t s^{3-2H}s^{3-2H}\left(\int_s^t (x-s)^{H-3/2}(2s)^{H-3/2}dx\right)^2 ds \quad (16)$$

$$= \frac{\pi}{2\Gamma^2(H-1/2)(H-1/2)^2}\int_0^t (t-s)^{2H-1}ds$$

$$= \frac{\pi t^{2H}}{4H\Gamma^2(H-1/2)(H-1/2)^2} .$$

On the other hand, by virtue of (16) and the property of the Gamma function, we get

$$c_H{}^2\int_0^t n_H{}^2(t,s)ds \le \frac{\Gamma(1+2H)t^{2H}}{4H\Gamma^2(H-1/2)(H-1/2)^2} = \frac{\Gamma(2H)t^{2H}}{2\Gamma^2(H-1/2)(H-1/2)^2} = \frac{\Gamma(2H-1)t^{2H}}{\Gamma^2(H-1/2)(H-1/2)^2} = \frac{\Gamma(2H-1)t^{2H}}{\Gamma(H-1/2)\Gamma(H-1/2)} . \quad (17)$$

Thus, applying the assumptions on A and (17) we can obtain that

$$E(I_3(t)) \le \frac{\Gamma(2H-1)}{(1-\varepsilon)\Gamma(H-1/2)\Gamma(H-1/2)} M^2T^{2H}\tilde{\sigma}^2\|h\|^2_{L([0,t];U)} . \quad (18)$$

Combining (14), (15) and (18), we get

$$E\|X(t)-Y(t)\|_U^2 \le \frac{2TM^2C_f}{(1-\varepsilon)^2}\int_0^t E\|X(s)-Y(s)\|_U^2 ds + \frac{2\Gamma(2H-1)}{(1-\varepsilon)^2\Gamma(H-1/2)\Gamma(H-1/2)} M^2T^{2H}\tilde{\sigma}^2\|h\|^2_{L([0,t];U)} .$$

Then, Gronwall's lemma implies that for any $t>0$,

$$E\|X(t)-Y(t)\|_U^2 \le \frac{2\Gamma(2H-1)}{(1-\varepsilon)^2\Gamma(H-1/2)\Gamma(H-1/2)} M^2T^{2H}\tilde{\sigma}^2 \int_0^t e^{\frac{2TM^2C_f{}^2(t-s)}{(1-\varepsilon)^2}}\|h(s)\|_U^2 ds \cdot$$

Hence, we may write that

$$d_\infty^2(X,Y) \le \frac{2\Gamma(2H-1)}{(1-\varepsilon)^2\Gamma(H-1/2)\Gamma(H-1/2)} M^2T^{2H}\tilde{\sigma}^2 e^{\frac{2TM^2C_f{}^2}{(1-\varepsilon)^2}}\int_0^T \|h(s)\|_U^2 ds$$

and

$$[W_2^{d_\infty}(Q,P_\varphi)]^2 \le 2C_{T,H}H\left(Q\middle|P_\varphi\right)$$

with $C_{T,H} = \frac{2\Gamma(2H-1)}{(1-\varepsilon)^2\Gamma(H-1/2)\Gamma(H-1/2)} M^2T^{2H}\tilde{\sigma}^2 e^{\frac{2TM^2C_f{}^2}{(1-\varepsilon)^2}}$.

The proof is complete.

Remark 3.1 It is obvious that if T_2 -transportation cost inequality holds using the uniform distance, then T_2 -transportation cost inequality also holds using the L^2 distance. Hence, under the conditions of the Theorem 3.1, the probability measure P_φ satisfies $T_2(C)$ on the metric space $C([0,T];U)$ with $C = \frac{\Gamma(2H-1)}{(1-\varepsilon)^2\Gamma(H-1/2)\Gamma(H-1/2)} M^2T^{2H}\tilde{\sigma}^2 \frac{e^{\frac{2TM^2C_f{}^2}{(1-\varepsilon)^2}T}-1}{\frac{2TM^2C_f{}^2}{(1-\varepsilon)^2}}$ when using the metric

$$d_2(r_1,r_2) = \left(\int_0^T \|r_1(t)-r_2(t)\|_U^2 dt\right)^{1/2}, \quad r_1,r_2 \in C([0,T];U) .$$

4. An Example

Let $K = L^2(0,\pi)$ and $e_n = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n \in N$. Then $\{e_n\}$ is a complete orthonormal basis in K. Let $U = L^2(0,\pi)$ and $A = \frac{\partial^2}{\partial x^2}$ with domain $D(A) = L_0^1(0,\pi)\cap L^2(0,\pi)$. Then, it is well known that $Au = -\sum_{n=1}^{\infty} n^2\langle u,e_n\rangle_U e_n$ for any $u \in U$, and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t): U \to U$, where $S(t)u = \sum_{n=1}^{\infty} e^{-n^2t}\langle u,e_n\rangle_U e_n$. This gives that $\|S(t)\| \le e^{-t}$.

Furthermore, for each $y \in U$, $(-A)^{-1/2}y = \sum_{n=1}^{\infty}\frac{1}{n}\langle u,e_n\rangle_U e_n$.

The operator $A^{-1/2}$ is given by

$$(-A)^{1/2}y=\sum_{n=1}^{\infty}n\langle u,e_n\rangle_U e_n$$

on the space $D((-A)^{1/2})=\left\{y(\cdot)\in U,\sum_{n=1}^{\infty}n\langle y,e_n\rangle_U e_n\right\}$. In particular, $\|A^{-1/2}\|=1$.

In order to define the operator $Q:K\to K$, we choose a sequence $\{\sigma_n\}_{n\ge1}\subset R^+$ and set $Qe_n=\sigma_n e_n$, and assume that $tr(Q)=\sum_{n=1}^{\infty}\sqrt{\sigma_n}<\infty$. Define the process $S_Q^H(t)$ by $S_Q^H(t)=\sum_{n=1}^{\infty}\sqrt{\sigma_n}S_n^H(t)e_n$, where $H\in(1/2,1)$ and $\{S_n^H\}_{n\in N}$ is a sequence of two-sided one-dimensional sub-fractional Brownian motions mutually independent.

Then we consider the following neutral stochastic equation:

$$\begin{cases} d[u(t,x)-\beta\sin u(t,x(t-\tau)(\cos(t)+\cos(\sqrt{2t}))] \\ =[\frac{\partial^2}{\partial x^2}u(t,x)+\beta u(t,x(t-\tau)(\cos(t)+\cos(\sqrt{2t}))]dt \\ +\sigma(t)dS_Q^H(t),t\in[0,T],x\in[0,\pi] \\ \qquad u(t,0)=u(t,\pi)=0,t\in[0,T], \end{cases}$$

where $\tau>0$ and the initial condition $u(t,x)=\varphi(t,x),t\in[-\tau,0],x\in[0,\pi]$.

Take

$$f(t,\phi_t)(\eta)=\beta[\phi(\eta_t)(\sin(t)+\sin(\sqrt{2t}))]$$

$$G(t,\phi_t)(\eta)=\beta[\sin(\phi(\eta_t))(\cos(t)+\cos(\sqrt{2t}))].$$

Thus, one has

$$\|f(t,x)-f(t,y)\|_U^2\le4\beta^2\|x-y\|_C^2$$

$$\|f(t,x)\|_U^2\le4\beta^2(1+\|x\|_C^2)$$

$$\|(-A)^{1/2}G(t,x)-(-A)^{1/2}G(t,y)\|\le4\beta^2\|x-y\|_C^2.$$

If P_φ is the law of the solution process of Eq. (5.1). Then, the probability measure P_φ satisfies $T_2(C)$ on the metric space $C([0,T];U)$ with the metric d_∞, provided that, $4\beta^2(1+T^{1/2}+1)<1$ and $\sigma(t)$ satisfies the conditions in the Theorem 3.1, according to the Theorem 3.1.

5. Conclusion

In this paper, by using the Girsanov transformation argument, we establish the quadratic transportation inequalities for the law of the mild solution of stochastic neutral partial differential equations of retarded type driven by sub-fractional Brownian motion with Hurst parameter H>1/2 under the d_2 metric and the uniform metric d_∞. In our furture work, we will explore the quadratic transportation inequalities for the law of the solution of stochastic neutral partial differential equations of retarded type driven by sub-fractional Brownian motion with Hurst parameter H<1/2.

References

[1] Djellout, H, Guillin A, Wu, L: Transportation cost-information inequalities for random dynamical systems and diffsions. *Ann. Probab.* 32, 2702-2732 (2004).

[2] Wu, L, Zhang, Z: Talagrand's T_2-transportation inequality w.r.t. a uniform metric for diffusions. *Acta Math. Appl. Sin. Engl. Ser.* 20, 357-364 (2004).

[3] Wu, L, Zhang, Z: Talagrand's T_2-transportation inequality and log-Sobolev inequality for dissipative SPDEs and applications to reaction-diffusion equations. *Chinese Ann. Math. Ser. B.* 27, 243-262 (2006).

[4] Pal, S: Concentration for multidimensional diffusions and their boundary local times. Probab. *Theory Relat. Fields.* 154, 225-254 (2012).

[5] Ustunel, A. S: Transport cost inequalities for diffusions under uniform distance. *Stochastic Analysis and Related Topics*. 22, 203-214 (2012).

[6] Wang F. Y: Transportation cost inequalities on path spaces over Riemannian manifolds. *Illinois J. Math.* 46, 1197-1206 (2002).

[7] Wang F. Y: Probability distance inequalities on Riemannian manifolds and path spaces. *J. Funct. Anal.* 206, 167-190 (2004).

[8] Wu, L: Transportation inequalities for stochastic differential equations of pure jumps. *Ann. Inst. Henri Poincare Probab. Stat.* 46, 465-479 (2010).

[9] Mendy, I: A Stochastic differential equation driven by a sub-fractional Brownian motion. (2010) Preprint.

[10] Bao, J, Wang, F. Y, Yuan, C: Transportation cost inequalities for neutral functional Stochastic equations. *Zeitschrift Fur Analysis Und Ihre Anwendungen*. 23(4), 457-475 (2013).

[11] Saussereau, B: Transportation inequalities for stochastic differential equations driven by a fractional Brownian motion. *Bernoulli.* 18(1), 1-23 (2012).

[12] Li, Z, Luo, J. W: Transportation inequalities for stochastic delay evolution equations driven by fractional Brownian motion. *Front. Math. China*. 10(2), 303-321 (2015).

[13] Bojdecki, T. L. G, Gorostiza, L. G, Talarczyk, A: Sub-fractional Brownian motion and its relation to occupation times. *Statistics and Probability Letters*. 69, 405-419 (2004).

[14] Bojdecki, T. L. G, Gorostiza, L. G, Talarczyk, A: Limit theorems for occupation time fluctuations of branching systems 1: Long-range dependence. *Stochastic Processes and their Applications*. 116, 1-18 (2006).

[15] Bojdecki, T. L. G, Gorostiza, L. G, Talarczyk, A: Some extensions of fractional Brownian motion and sub-fractional Brownian motion related to particle systems. *Electronic Communications in Probability.* 12, 161-172 (2007).

[16] Tudor, C: Some properties of the sub-fractional Brownian motion. *Stochastics*. 79, 431-448 (2007).

[17] Tudor, C: Some aspects of stochastic calculus for the sub-fractional Brownian motion. *Analele Universitatii Bucuresti, Mathematica*. 199-230 (2008).

[18] Tudor, C: Inner product spaces of integrands associated to sub-fractional Brownian motion. *Statistics and Probability Letters*. 78, 2201-2209 (2008).

[19] Tudor, C: Sub-fractional Brownian Motion as a Modelin Finance. University of Bucharest (2008).

[20] Tudor, C: On the Wiener integral with respect to a sub-fractional Brownian motionon an interval. *Journal of Mathematical Analysis and Applications*. 351, 456-468 (2009).

[21] Tudor, C: Berry-Esseen bounds and almost sure CLT for the quadratic variation of the sub-fractional Brownian motion. *Journal of Mathematical Analysis and Applications*. 375, 667-676 (2011).

[22] Yan, L, Shen, G: On the collision local time of sub-fractional Brownian motions. *Statistics and Probability Letters*. 80, 296-308 (2010).

[23] Dzhaparide, K, Van Zanten, H: A series expansion of fractional Brownian motion. *Probab. Theory Relat. Fields*. 103, 39-55 (2004).

[24] Mendy, I: Parametric estimation for sub-fractional Ornstein-Uhlenbeck process. *Journal of Statistical Planning and Inference*. 143, 663-674 (2013).

[25] Nualart, D: The Malliavin Calculus and Related Topics, 2nd edn, Springer-Verlag (2006).

[26] Pazy, A: Semigroup of linear operators and applications to partial differential equations. Spring Verlag, New York. (1992)

[27] Li, Z, Zhou, G. L, Luo, J. W: Stochastic delay evolution equations driven by sub-fractional Brownian motion. *Advances in Difference Equations.* 2015 (48), DOI 10.1186/s13662-015-0366-1 (2015).

[28] Da Prato, G, Zabczyk, J: Stochastic Equations in Infinite Dimensions, Cambridge University Press (1992).