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Asymptotic Analytical Solutions of an Electrostatically Actuated Microbeam Base on Homotopy Analysis Method

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Abstract

Presented herein is an analytical approach based on homotopy analysis method (HAM) used to deal with the seventh-order Duffing type problem with high-order nonlinearity. Such a problem corresponds to the large-amplitude vibration of an electrostatically actuated microbeam. Unlike tradition HAM, the convergence-control auxiliary parameters $\hbar_i\left(i=1,2\cdots m\right)$ are introduced in the present approximation to improve the accuracy. To verify the efficient of present approach, illustrative examples are provided and compared between the results obtained by analytical and numerical method.

Keywords

Homotopy Analysis Method, Electrostatically Actuated Microbeam, Convergence-Control Parameter

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1. Introduction

Micro-electro-mechanical systems (MEMS) are batch-fabricated devices and structures at a micro-scale level [1]. Compared to the traditional mechanical systems, the MEMS devices have attracted intensive research and fully developed in the past decade with varieties of applications in different disciplines, such as aerospace, optical and biomedical engineering [2], because of its low power consumption, small size and high reliability. The sensors and actuators are potential applications of such systems, which are used in different engineering applications. Ghanbari et al. [3] adopted data fusion technique to study the delay compensation of tilt sensors based on MEMS accelerometer. A microstacked PZT actuator of dimensions 8 mm × 0.8 mm × 0.4 mm and capable of 2.3-μm actuation under a voltage of 100 V was fabricated and characterized in [4] by Sabri et al.. The archetypal electrostatic micro-switch can be modeled by an electrostatically driven microbeam and a pair of fixed eletrodes. Vahabisani et al. [5] revealed a monolithic wafer-level MEMS waveguide switch for millimeter-wave application. Xi et al. [6] showed the simulation and visual test contact progress of a MEMS inertial switch flexible electrode. Theoretically, most of the nonlinear oscillation problems arising in the microbeam based on MEMS are governed by a set of differential equations and auxiliary conditions which arise from modeling processes [7]. For instance, Pu et al. [8] proposed an electrothermal-driven gap adjustable MEMS comb structure and derived the mechanical-electrical coupling equation of the system. Simulation of the micromodel was carried out with SPICE. In contrast with the numerical analysis, in order to better investigate nonlinear differential

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equations, the quest of accurate analytical or semi-analytical techniques is highly indispensable. Since the scientific merit of the analytical solutions provided a superior understanding and insight of the interrelationship between the governing parameters and resulting solutions, such that the availability of the design practices in science and engineering can be advanced and benefited.

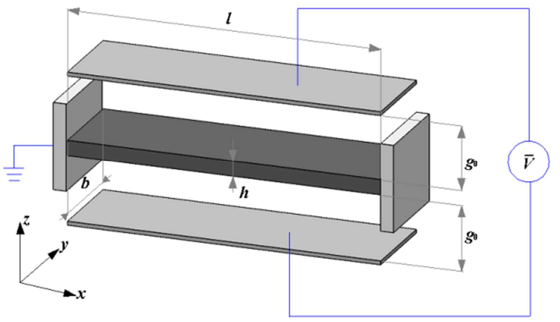


Figure 1. Geometry for an electrostatically actuated microbeam with fixed supports.

Recently, some approximate methods are considered to be the powerful methods capable of dealing with strongly nonlinear behaviors, and can converge to an accurate periodic solution for smooth nonlinear systems, such as variational iteration method [9-12], the modified perturbation method [13-16], parameter-expansion method [17, 18], improved harmonic balance methods [19-22], maxemin approach [23-25] and energy balance method [26, 27]. Meanwhile, the homotopy analysis method (HAM) [28-30] was emerged as one of the robust and efficient analytical techniques in solving nonlinear problems. In order to better use it to tackle various nonlinear problems, Liao [31] investigated the so-called Blasius boundary-layer flows in fluid mechanics by the OHAM approach. Niu et al [32] put forward a one-step OHAM for nonlinear differential equations. Qian et al. [33] analyzed asymptotic analytical solutions using HAM for strongly nonlinear vibrations of a restrained cantilever beam with an intermediate lumped mass. Abbasbandy et al. [34] proposed a modified approach in the determining of the convergence-control parameter value. Mallory et al. [35] analyzed the control of error for solutions for the non-local whitham equation in HAM. Qian et al. [2] studied the nonlinear vibration of an electrostatically actuated microbeam. Moreover, the OHAM approach [2, 32] is introduced to accelerate the convergence of solutions. Thus, the prime objective of this paper is to explore the utility of an iterative approach based on HAM for the nonlinear oscillation problem arising in the MEMS microbeam model. Consider the seventh-order Duffing type problem with high-order nonlinearity, which is a fixed-fixed microbeam placed between two stationary electrodes with uniform thickness h, length l, width b $(b \ge 5h)$, g_0 is the initial gap and $\overline{V}$ is electrostatic load, as shown in Fig. 1. We obtained the equation of motion that governs the transverse deflection $w(x,t)$ as follows

$$\overline{E}I\frac{\partial^4 w}{\partial x^4}+\rho S\frac{\partial^2 w}{\partial t^2}=\left[\overline{N}+\frac{ES}{2l}\int_0^1(\frac{\partial w}{\partial x})^2\right]\frac{\partial^2 w}{\partial x^2}+q(x,t)\,, \qquad (1)$$

where $\overline{E}=E/\left(1-\nu^2\right)$, E, ν and ρ be effective modulus, Young's modulus, Poisson's ratio and density, respectively. $\overline{N}$ be the tensile or compressive axial load created by the mismatch of both thermal expansion coefficient and crystal lattice period between substrate and the thin film (microbeam). It is considered that $\overline{E}$ simply becomes the Young's modulus E, $S=bh$ and $I=bh^3/12$ are the area and moment of inertia of the cross-section, respectively. The driving force per unit length $q(x,t)$ is denoted as following equation, resulting from electrostatic excitation [36]

$$q(x,t)=\frac{\varepsilon_v bv}{2}\left[\frac{1}{(g_0-w)^2}-\frac{1}{(g_0+w)^2}\right], \qquad (2)$$

where ε_v is the dielectric constant of the gap madium which is usually taken as $\varepsilon_v=8.85pF/m$. The boundary conditions are given by

$$x=0,1:\ w=\frac{\partial w}{\partial x}=0\,, \qquad (3)$$

For convenience, we injected the following non-dimensional variables and parameters into the Eq. (1)

$$\xi=\frac{x}{l},W=\frac{w}{g_0},t=t\sqrt{\frac{\overline{E}I}{\rho bhl}},\alpha=6(\frac{g_0}{h})^2,N=\frac{\overline{N}l^2}{\overline{E}I},V^2=\frac{24\varepsilon_v l^4 v^2}{\overline{E}h^3 g_0^3}, \qquad (4)$$

We can gain the normalized form of the governing equation

$$\frac{\partial^4 W}{\partial\xi^4}+\frac{\partial^2 W}{\partial t^2}=\left[N+\alpha\int_0^1\left(\frac{\partial W}{\partial\xi}\right)^2\right]\frac{\partial^2 W}{\partial\xi^2}+\frac{V^2}{4}\left[\frac{1}{(1-W)^2}-\frac{1}{(1+W)^2}\right], \qquad (5)$$

with the non-dimensional boundary conditions

$$\xi=0,1:w=\frac{\partial w}{\partial\xi}=0\,. \qquad (6)$$

The deflection function $W(\xi,t)$ as the product of two separate functions as most

earlier work suggested [37,38]

$$W(\xi;t)=u(t)\,\varphi(\xi)\,. \qquad (7)$$

By Eq. (4), obviously, Eq. (7) satisfies the boundary conditions listed in Eq. (6). In order to avoid division by zero in the electrostatic force term, we first multiplied both sides of Eq. (5) by $\left(1-W^2\right)^2$

$$\left(1-W^2\right)^2\left(\frac{\partial^4 W}{\partial\xi^4}+\frac{\partial^2 W}{\partial t^2}\right)=\left(1-W^2\right)^2\left[N+\alpha\int_0^1\left(\frac{\partial W}{\partial\xi}\right)^2\right]\frac{\partial^2 W}{\partial\xi^2}+V^2W\,, \quad (8)$$

Then, substituting Eq. (7) into the Eq. (8), we get

$$\begin{aligned}&\ddot{u}\left(\varphi^5u^4-2\varphi^3u^2+\varphi\right)+\left(\varphi''''-N\varphi''-V^2\varphi\right)u\\&+\left(-2\varphi''''\varphi^2+2N\varphi''\varphi^2-\alpha\varphi''\int_0^1\left(\varphi'\right)^2d\xi\right)u^3\\&+\left(\varphi''''\varphi^4-N\varphi''\varphi^4+2\alpha\varphi''\varphi^2\int_0^1\left(\varphi'\right)^2d\xi\right)u^5\\&+\left(-\alpha\varphi''\varphi^4\int_0^1\left(\varphi'\right)^2d\xi\right)u^7=0\end{aligned}. \quad (9)$$

Finally, we multiply $\varphi(\xi)$ by Eq. (9) and integrate the outcome from 0 to 1, we obtain

$$\ddot{u}\left(a_1u^4+a_2u^2+a_3\right)+a_4u+a_5u^3+a_6u^5+a_7u^7=0\,, \quad (10)$$

with initial conditions

$$u(0)=A\,,\dot{u}(0)=0\,, \quad (11)$$

where

$$\begin{aligned}&a_1=\int_0^1\varphi^6d\xi\,,\ a_2=-2\int_0^1\varphi^4d\xi\,,\ a_3=\int_0^1\varphi^2d\xi\,,\\&a_4=\int_0^1\left(\varphi''''\varphi-N\varphi''\varphi-V^2\varphi^2\right)d\xi\,,\\&a_5=-\int_0^1\left(2\varphi''''\varphi^3-2N\varphi''\varphi^3+\alpha\varphi''\varphi\int_0^1\left(\varphi'\right)^2d\xi\right)d\xi\,,\\&a_6=\int_0^1\left(\varphi''''\varphi^5-N\varphi''\varphi^5+2\alpha\varphi''\varphi^3\int_0^1\left(\varphi'\right)^2d\xi\right)d\xi\,,\\&a_7=-\int_0^1\left(\alpha\varphi''\varphi^5\int_0^1\left(\varphi'\right)^2d\xi\right)d\xi\,,\end{aligned} \quad (12)$$

where a overdot $(\cdot)$ denotes the derivative with respect to the dimensionless time variable τ, while a prime $(')$ indicates the differentiation with respect to coordinate variable ξ. The function φ in Eqs. (12) can be substituted by $\varphi(\xi)=16\xi^2\left(1-\xi\right)^2$.

The paper is organized as follows. The methodology of an iterative approach based on HAM for solving equation of motion for the microbeam with high-order nonlinearity is described in Section 2. In Section 3, numerical comparisons are carried out to authenticate the accuracy and correctness of the present method. Conclusion remarks and recommendations are given in Section 4.

2. Analytical Method Based on HAM

We consider large-amplitude vibration of an electrostatically actuated microbeam, described by the nonlinear Eqs. (10) and (11). A new independent variable $\tau=\omega t$ is introduced and recast in Eqs. (10) and (11) as follows

$$\omega^2u''\left(a_1u^4+a_2u^2+a_3\right)+a_4u+a_5u^3+a_6u^5+a_7u^7=0\,, \quad (13)$$

$$u(0)=A\,,u'(0)=0\,, \quad (14)$$

where a prime denotes the derivative with respect to τ, and ω is the nonlinear frequency.

Generally, the oscillation of a conservative system without damping effect shows a periodic motion. So, in Eq. (13), the periodic solution can be represented by the following base function

$$\left\{\cos(2m-1)\tau\,|\,m=1,2,3,\cdots\right\}. \quad (15)$$

In order to satisfy the initial conditions in Eq. (14), the initial guess of $u(\tau)$ for the zeroth-order deformation equation is taken as

$$u_0(\tau)=A\cos\tau\,. \quad (16)$$

In the following, we expand $u(\tau)$ and ω in a series of the form.

$$u(\tau)=u_0(\tau)+\sum_{m=1}^{+\infty}u_m(\tau)\,, \quad (17)$$

$$\omega=\omega_0+\sum_{m=1}^{+\infty}\omega_m\,. \quad (18)$$

Constructing the following power series

$$u(\tau;q)=u_0(\tau)+\sum_{m=1}^{+\infty}u_m(\tau)q^m\,, \quad (19)$$

$$\omega(q)=\omega_0+\sum_{m=1}^{+\infty}\omega_mq^m\,, \quad (20)$$

where $q\in[0,1]$ is embedding parameters and

$$u_m(\tau)=\frac{1}{m!}\frac{\partial^m u(\tau;q)}{\partial q^m}\bigg|_{q=0}\,, \quad (21)$$

$$\omega_m=\frac{1}{m!}\frac{\partial^m\omega(q)}{\partial q^m}\bigg|_{q=0}\,. \quad (22)$$

According to Eq. (13), the nonlinear operator is written as

$$\begin{aligned}&N\left[u(\tau;q),\omega(q)\right]\\&=\left[\omega(q)\right]^2\frac{\partial^2 u(\tau;q)}{\partial\tau^2}\left(a_1\left[u(\tau;q)\right]^4+a_2\left[u(\tau;q)\right]^2+a_3\right)\\&+a_4u(\tau;q)+a_5\left[u(\tau;q)\right]^3+a_6\left[u(\tau;q)\right]^5+a_7\left[u(\tau;q)\right]^7\end{aligned} \quad (23)$$

The auxiliary linear operator of a conservative system is given by

$$L\left[u(\tau)\right]=\omega_0^2\left(\frac{d^2u(\tau)}{d\tau^2}+u(\tau)\right). \quad (24)$$

The auxiliary linear operator L is chosen in such a way that all solutions of the corresponding high-order formation equations exist and can be expressed by the general form of the base function.

We consider an iterative method with respect to the embeding convergence control parameter $\hbar_m,(m=1,2,\cdots)$ of the form

$$L\left[u_m(\tau)-\chi_m u_{m-1}(\tau)\right]=\hbar_m R_m\left(u^{m-1},\omega^{m-1}\right), \quad (25)$$

with the initial conditions

$$u_m(0)=0\,,u'_m(0)=0\ (m\geq 1), \quad (26)$$

in which

$$u^n=\left\{u_1(\tau),u_2(\tau),\cdots,u_n(\tau)\right\},\ \omega^n=\left\{\omega_0,\omega_1,\cdots,\omega_n\right\}. \quad (27)$$

$$\chi_m=\begin{cases}0,\ m\leq 1\\1,\ m>1\end{cases}, \quad (28)$$

$$R_m\left(u^{m-1},\omega^{m-1}\right)=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}N\left[u(\tau;q)\right]}{\partial q^{m-1}}\right|_{q=0}. \quad (29)$$

Suppose that $\hbar_m,(m=1,2,\cdots)$ are properly chosen, the power series solutions in Eqs. (19) and (20) converge at $q=1$, thus we get the $u(\tau)$ and ω from the Eqs. (17) and (18)

It is found that R_m can also be written as

$$R_m\left(u^{m-1},\omega^{m-1}\right)=\sum_{k=1}^{\phi(m)}d_k\left(\omega^{m-1}\right)\cos(2k-1)\tau\,, \quad (30)$$

where d_k is an function of ω^{m-1} and $\phi(m)$ is an integer that depends on m.

Following the rule of solution expression and the linear operator L, the terms of $\cos\tau$ must be vanished in Eq. (25). Otherwise, the so-called secular terms, such as $\tau\cdot\sin\tau$, are present in the periodic solutions. To avoid the presence of such terms, their coefficients are set to zero as follows

$$\frac{1}{\pi}\int_0^{2\pi}\left[\hbar_m R_m\left(u^{m-1},\omega^{m-1}\right)\right]\cos\tau\,d\tau=0\,. \quad (31)$$

The solutions of ω^{m-1} ($m=1,2,\cdots$) in Eqs. (25) and (31) can be successively calculated. The periodic solutions are obtained from

$$\begin{aligned}u_m(\tau)=\chi_m u_{m-1}(\tau)+\frac{\hbar_m}{\omega_0^2}\sum_{k=2}^{\phi(m)}\frac{d_k\left(\omega^{m-1}\right)\cos(2k-1)\tau}{\left(1-(2k-1)^2\right)}\\+C_1\cos\tau+C_0\end{aligned}, \quad (32)$$

in which C_0 and C_1 are constants that can be determined by using the initial condition given in Eq. (26). Obviously, for any $u_m(\tau)$, $C_0=0$.

Hence, the mth-order approximations are defined as

$$u(\tau)=\sum_{i=0}^{m}u_i(\tau)\,, \quad (33)$$

$$\omega=\sum_{i=0}^{m}\omega_i\,. \quad (34)$$

For $m=1$, one obtains ω_0 from Eq. (31) as follows:

$$\omega_0=\sqrt{\frac{64a_4+48a_5A^2+40a_6A^4+35a_7A^6}{40a_1A^4+48a_2A^2+64a_3}}\,. \quad (35)$$

Using Eq. (32) gives rise to

$$\begin{aligned}u_1(\tau)=&\left(\frac{b_{1,3}\hbar_1}{8\omega_0^2}+\frac{b_{1,5}\hbar_1}{24\omega_0^2}+\frac{b_{1,7}\hbar_1}{48\omega_0^2}\right)\cos\tau-\frac{b_{1,3}\hbar_1}{8\omega_0^2}\cos 3\tau\\&-\frac{b_{1,5}\hbar_1}{24\omega_0^2}\cos 5\tau-\frac{b_{1,7}\hbar_1}{48\omega_0^2}\cos 5\tau\end{aligned} \quad (36)$$

where

$$b_{1,3}=\frac{1}{4}A^3a_5+\frac{5}{16}A^5a_6+\frac{21}{64}A^7a_7-\frac{5}{16}A^5a_1\omega_0^2-\frac{1}{4}A^3a_2\omega_0^2\,, \quad (37)$$

$$b_{1,5}=\frac{1}{16}A^5a_6+\frac{7}{64}A^7a_7-\frac{1}{16}A^5a_1\omega_0^2\,, \quad (38)$$

$$b_{1,7}=\frac{1}{64}A^7a_7\,. \quad (39)$$

For m = 2, Eqs. (16), (35) and (36) are substituted into Eq. (31), we have

$$\begin{aligned}\omega_1 = &\frac{\hbar_1}{384A(5A^4a_1+6A^2a_2+8a_3)\omega_0^3}(192a_4b_{1,3}+288A^2a_5b_{1,3}+300A^4a_6b_{1,3}\\&+294A^6a_7b_{1,3}+64a_4b_{1,5}+144A^2a_5b_{1,5}+180A^4a_6b_{1,5}+196A^6a_7b_{1,5}+32a_4b_{1,7}\\&+72A^2a_5b_{1,7}+100A^4a_6b_{1,7}+119A^6a_7b_{1,7}+180A^4a_1b_{1,3}\omega_0^2+96A^2a_2b_{1,3}\omega_0^2\\&-192a_3b_{1,3}\omega_0^2-84A^4a_1b_{1,5}\omega_0^2-144A^2a_2b_{1,5}\omega_0^2-64a_3b_{1,5}\omega_0^2\\&-100A^4a_1b_{1,7}\omega_0^2-72A^2a_2b_{1,7}\omega_0^2-32a_3b_{1,7}\omega_0^2)\end{aligned} \tag{40}$$

Computing Eq. (30) for $R_2\left(\mathrm{u}^1,\omega^1\right)$ deduces

$$\begin{aligned}R_2\left(\mathrm{u}^1,\omega^1\right) &= b_{2,3}\cos 3\tau + b_{2,5}\cos 5\tau + b_{2,7}\cos 7\tau\\&+ b_{2,9}\cos 9\tau + b_{2,11}\cos 11\tau + b_{2.13}\cos 13\tau\end{aligned} \tag{41}$$

To save space, the coefficients $b_{2,i}$ $(i=3,5,\cdots,13)$ are not presented herein, but they can be readily derived using Eq. (31). In addition, $u_2(\tau)$ is given by

$$\begin{aligned}u_2(\tau) &= u_1(\tau)\\&+\left(\frac{b_{2,3}\hbar_2}{8\omega_0^2}+\frac{b_{2,5}\hbar_2}{24\omega_0^2}+\frac{b_{2,7}\hbar_2}{48\omega_0^2}+\frac{b_{2,9}\hbar_2}{80\omega_0^2}+\frac{b_{2,11}\hbar_2}{120\omega_0^2}+\frac{b_{2,13}\hbar_2}{168\omega_0^2}\right)\cos\tau\frac{b_{2,3}\hbar_2}{8\omega_0^2}\cos 3\tau\\&-\frac{b_{2,5}\hbar_2}{24\omega_0^2}\cos 5\tau-\frac{b_{2,7}\hbar_2}{48\omega_0^2}\cos 7\tau-\frac{b_{2,9}\hbar_2}{80\omega_0^2}\cos 9\tau-\frac{b_{2,11}\hbar_2}{120\omega_0^2}\cos 11\tau-\frac{b_{2,13}\hbar_2}{168\omega_0^2}\cos 13\tau\end{aligned} \tag{42}$$

According to Eqs. (33) and (34), the corresponding second-order analytical approximation for Eq. (13) is

$$u(\tau) = u_0(\tau) + u_1(\tau) + u_2(\tau), \tag{43}$$

where

$$\omega = \omega_0 + \omega_1 . \tag{44}$$

The higher-order approximations for ω and $u(t)$ can be established in a similar manner.

To measure the accuracy of the mth-order approximation, the squared residual error of the mth-order analytical approximation can be defined as

$$\Delta_m = \int_0^{2\pi}\left(N\left[u(\tau)\right]\right)^2 d\tau . \tag{45}$$

For the sake of computational efficiency, according to the definition of definite intergral, the squared residual error Δ_m is calculated numerically

$$\Delta_m \approx \tilde{\Delta}_m = \frac{2i\pi}{M}\sum_{i=1}^{m}\left(N\left[u\left(\frac{2i\pi}{M}\right)\right]\right), \tag{46}$$

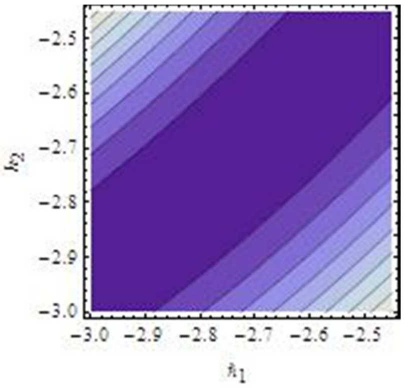
where M is an integer. In this paper, $M=50$ is used. Given the initial guess $u_0(\tau)$ and the auxiliary linear operator L, the discrete squared residual error $\tilde{\Delta}_m$ is only dependent on the convergence-control parameters $\hbar_i$ $(i=1,2,\cdots m)$ whose optimal values are determined by the minimum of $\tilde{\Delta}_m$. By minimizing the residual error $\tilde{\Delta}_m$, the corresponding value of the convergence-control parameter $\hbar_i$ at the given order of analytical approximation m can be optimized and decided.

3. Illustrative Example and Discussion

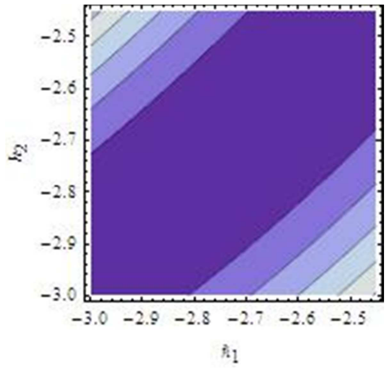
In order to validate the effectiveness of the current approach, the four cases are analyzed. The approximation at each iteration and the corresponding discrete square residual for various parameters N, α, V and amplitudes A are listed in Table 1 for four cases. The values of dimensionless parameters N, α, V in Table 1 correspond to different forces and voltages acting on the microbeam. It shows that the square residual of the approximation decreases as the iteration times m increases. Table 2 reveals the nonlinear frequency in [2], in this paper and exact solution ω_{ex} for the four modes. We notice that the nonlinear frequency ω_{HAM} in [2] much more close to ω_{ex} than in this paper as $m\to\infty$. But there is a insuperable defect of the OHAM in [2] which we need to obtain the exact solution ω_{ex}. Fig. 2 displays the selection of optimal value of $\hbar_1,\hbar_2$ for the cases given in Table 1. In Figs. 3-6, the phase portrait diagrams and time history responses for four cases governed by different amplitudes of vibration are depicted. We can clearly observe that the second-order approximation solutions are in good agreement with the numerical integral solutions.

Table 1. Comparision the square residual of the iterative approach and $\hbar=-1$ corresponding to four cases.

Mode	N	α	V	A	$\hbar_1$	$\hbar_2$	$\tilde{\Delta}_1$	$\tilde{\Delta}_2$	$\tilde{\Delta}_2$ $(\hbar=-1)$
1	10	24	0	0.3	-2.61321	-2.67752	0.11308	0.01289	6.15261
2	10	24	20	0.3	-2.60622	-2.66742	0.03810	0.00232	1.72671
3	10	24	10	0.6	-2.92186	-3.13442	25.0554	2.53801	176.840
4	10	24	20	0.6	-2.76198	-2.89511	0.98416	0.03208	13.5844



Mode 1



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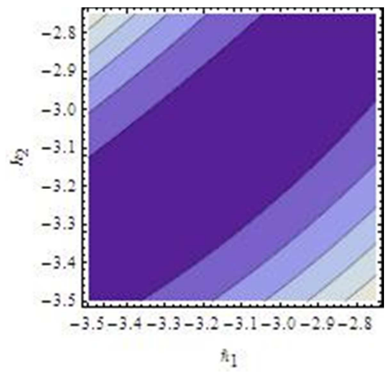

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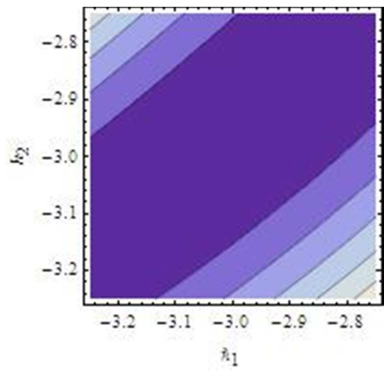

Mode 4

Figure 2. The selection of optimal convergence-control parameters $\hbar_1, \hbar_2$ for cases 1-4 in Table 1.

Table 2. Comparison of the exact approximate frequencies ω_{HAM} in [2], ω in this paper and the exact frequencies ω_{ex} corresponding to four cases for $m = 2$.

Mode	N	α	V	A	ω_{HAM}	ω	ω_{ex}
1	10	24	0	0.3	26.8372	26.8462	26.8372
2	10	24	20	0.3	16.6486	16.6546	16.6486
3	10	24	10	0.6	28.5368	28.6037	28.5382
4	10	24	20	0.6	18.5902	18.5947	18.5902

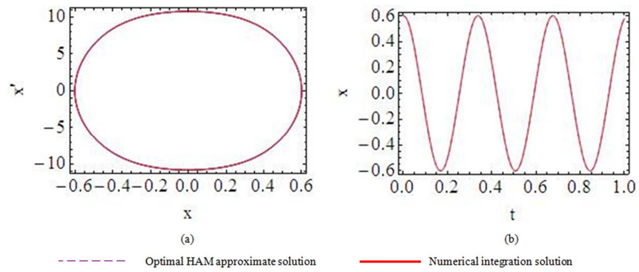

Figure 3. Comparison of the approximate and exact solutions for $A = 0.3$, $N = 10$, $\alpha = 24$ and $V = 0$, (a) Phase curve (b) Time history response.

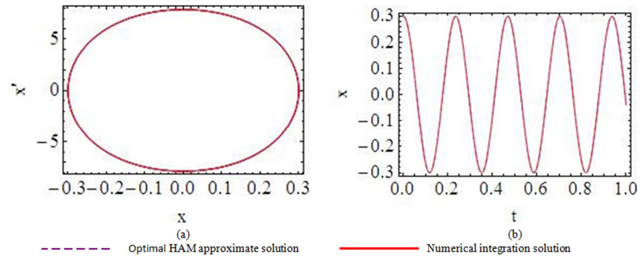

Figure 4. Comparison of the approximate and exact solutions for $A = 0.3$, $N = 10$, $\alpha = 24$ and $V = 20$, (a) Phase curve (b) Time history response.

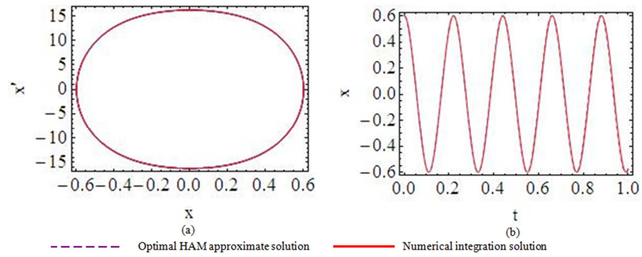

Figure 5. Comparison of the approximate and exact solutions for $A = 0.6$, $N = 10$, $\alpha = 24$ and $V = 10$, (a) Phase curve (b) Time history response.

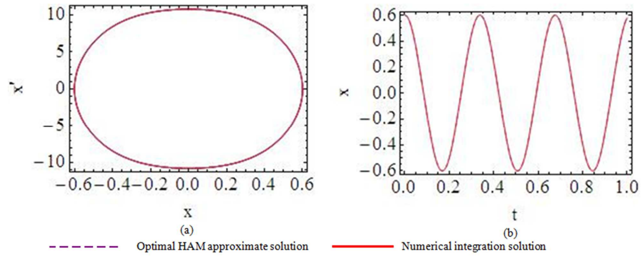

Figure 6. Comparison of the approximate and exact solutions for $A = 0.6$, $N = 10$, $\alpha = 24$ and $V = 20$, (a) Phase curve (b) Time history response.

4. Conclusions

In this paper, the nonlinear vibration equation arising in the microbeam based MEMS is used to reveal the solution approach for highly nonlinear terms problems. Applying the iterative approach, the nonlinear periodic motions of various vibration modes for the free vibration of microbeam are offered. Obviously, HAM is a special case of the iterative approach, when Eq. (25) rewrites as $L\left[u_m(\tau)-\chi_m u_{m-1}(\tau)\right]=\hbar R_m\left(u^{m-1},\omega^{m-1}\right)$. But, in order to minimize the squared residual error Δ_m, we do not confirm the nonlinear frequency ω_{HAM} approches to ω_{ex} as $m\to\infty$.

In an effort to extend the accuracy of the present approach, on the choice of auxiliary linear operator in the OHAM of the Cahn-Hilliard initial value problem is considered in order to find the best operator [39]. Much more OHAM based on the auxiliary linear operator L and auxiliary function $H(t)$ is strongly suggested to investigate later, if possible. Besides, it is significant to prove the existence of solution on Eq.(13), so we ought to try to do it in the future. Perhaps we can make it referencing to [40].

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Authors' Contributions

All the authors contribute equally and significantly in writing this paper. All the authors read and approve the final manuscript.

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