

"Restricted Euler Equations" Model Is Not Very Suitable for Revealing Properties Inherent to Euler Equations

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Abstract

For the restricted Euler equations, we prove the following: 1) If there is a time instant such that the perfect vector alignment between the vorticity vector and the strain matrix eigenvector happens at some point inside the 3D incompressible restricted Euler flow, it continues then permanently and keeps forever in the sense that one happens successively at all points which belong to the given point trajectory generated by the flow. 2) If the aforementioned trajectory exists, then, depending on initial data, one can either blow up for a finite time or not. What is interesting in doing so is that the aforesaid finite-time blow ups can be various and very different. In particular, we found the blow up such that the strain matrix eigenvalues all go at infinity (which can be both positive and negative) whereas the vorticity remains bounded at the same time. Such kind the vorticity behavior should be considered, generally speaking, as slightly unexpected if to take into account the well-known Beale-Kato-Majda (abbr. BKM) criterion for solutions of the genuine Euler equations to blow up at a finite-time. On the other hand, there are solutions to the restricted Euler equations which blow up at a finite time by a scenario similar to that is described in the BKM criterion. Summarizing all what is the aforesaid we see that the restricted Euler equations are not suitable enough for revealing properties inherent to the Euler equations.

Keywords

Finite-Time Blow up, the Euler Equations, the Restricted Euler Equations, Alignment, Vorticity, Strain, Vortex Stretching

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1. Introduction

The problem of global unambiguous solvability for the 3D Euler equations of ideal incompressible fluid as well as that of whether the 3D incompressible Euler equations solution can develop a finite time singularity from smooth initial data remain the most challenging open problems in the ideal incompressible fluid mathematical theory. Their main difficulty consists in that a dependence of pressure in the fluid flow on its velocity is non-local. In order to overcome this difficulty, a so-called model of the "restricted Euler equations" is proposed. (For the model definition, see the following paper text.) In this model, the original pressure term of the Euler equations is changed by a certain local

expression. The model idea goes back to Viellefosse [1, 2], Novikov [3] and Cantwell [4]. Later, the given model is studied in [5, 6, 7] by Liu, Tadmor and Wei. This paper prolongs exploration of the restricted Euler equations properties which was begun in [11].

2. Preliminaries

We start of the Cauchy problem for the 3D incompressible Euler equations in \mathbb{R}^3 although results presented below all remain valid for such of the kind flows inside open both bounded and unbounded subsets of \mathbb{R}^3 having smooth boundary as well as for the periodic flows in a torus T^3 . So, we consider the following problem:

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$$\frac{D\bar{u}}{Dt} = -\nabla p, \quad \text{div } \bar{u} = 0, \quad \bar{u}(\bar{x}, 0) = \bar{u}_o(\bar{x}), \quad (1)$$

where $\frac{D}{Dt} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + (\bar{u} \cdot \nabla)$ is the material derivative,

$\nabla \stackrel{\text{def}}{=} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\}$ is the gradient operator with respect to

\bar{x} , \bar{u}_o is the given initial flow velocity satisfying $\text{div } \bar{u}_o = 0$.

Here and in what follows 3D vectors and scalars are designated by bold letters and ordinary ones respectively.

That is $\bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\bar{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ and

$p = (p) \in \mathbb{R}^1$. As a rule, we shall omit the designation on the

space-time co-ordinates if those are designated by (\bar{x}, t) and

such the omission will not lead to misunderstanding, and, on

the contrary, we shall point out such the dependence

otherwise. For the given velocity $\bar{u}(\bar{x}, t)$ and pressure

$p(\bar{x}, t)$ let us introduce the 3×3 matrices

$$U \equiv U_{k,j} \stackrel{\text{def}}{=} \frac{\partial u_k}{\partial x_j},$$

$$U^T \equiv U_{j,k}, \quad S \equiv S_{k,j} \stackrel{\text{def}}{=} \frac{1}{2} \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right),$$

$$A \equiv A_{k,j} \stackrel{\text{def}}{=} \frac{1}{2} \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right),$$

$$P \equiv P_{k,j} \stackrel{\text{def}}{=} \frac{\partial^2 p}{\partial x_k \partial x_j},$$

$$E \equiv E_{k,j} \stackrel{\text{def}}{=} \delta_j^k,$$

where $k, j = 1, 2, 3$, and δ_j^k is the Kronecker delta symbol, i.e.

$\delta_j^k = 1$ when

$j = k$ and $\delta_j^k = 0$ otherwise. We recall that $A_{k,j}$ is related to the vorticity

$\bar{\omega} \stackrel{\text{def}}{=} \text{curl } \bar{u} = \nabla \times \bar{u}$ by means of the formulas

$$A_{k,j} = -\frac{1}{2} \sum_{i=1}^3 \varepsilon_{kji} \omega_i, \quad \omega_k = -\frac{1}{2} \sum_{i=1}^3 \varepsilon_{kji} A_{j,i},$$

where ε_{kji} is the Eddington alternating skew-symmetric

tensor with the normalization $\varepsilon_{123} = 1$. Other designations we

use are following. Operations of a vector (external) product

and scalar (inner) one are designated by the signs " \times " and

" \cdot " respectively. Thus, for any \bar{v} and \bar{w} ,

$$\bar{v} \times \bar{w} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1), \quad \bar{v} \cdot \bar{w} = \sum_{i=1}^3 v_i w_i.$$

We shall denote the vector $\left(\sum_{j=1}^3 M_{k,j} v_j \right)_{k=1}^3$ by $M \bar{v}$.

In what follows we shall use notions of the fluid particle trajectory, the Lagrangian particle marker, the Lagrangian variable also. Let us set

$$\bar{X} = (X_1, X_2, X_3) \in \mathbb{R}^3, \quad \bar{a} = (a_1, a_2, a_3) \in \mathbb{R}^3.$$

Definition. The function of $\bar{X}(\bar{a}, t)$ is said to be the fluid particle trajectory of the flow \bar{u} if it is a unique solution of the following differential equation:

$$\frac{d\bar{X}(\bar{a}, t)}{dt} = \bar{u}(\bar{X}(\bar{a}, t), t), \quad \bar{X}(\bar{a}, 0) = \bar{a}, \quad (2)$$

where $\bar{u}(\bar{x}, t)$ is the classical solution of Eq.(1) such that

$$\bar{u}(\bar{x}, t)|_{t=0} = \bar{u}_o(\bar{x}).$$

The pair (\bar{a}, t) is called the Lagrangian variables. Here \bar{a} is

the Lagrangian particle marker and t is time. A totality of all

the solutions of Eq.(2) generates a fluid particle trajectory

mapping $\bar{a} \rightarrow \bar{X}(\bar{a}, t)$. Since transition from the Euler fluid

flow discription to the Lagrangian one results in changing of

the variable \bar{x} on the variable $\bar{X}(\bar{a}, t)$, it is suitable $\bar{X}(\bar{a}, t)$

to call the Lagrangian variable also. In what follows we shall

assume always that $\bar{u}(\bar{x}, t)$ and $\bar{X}(\bar{a}, t)$ are considered for the

maximal time interval of existence of the classic (strong)

solution of Eq.(1) and shall not indicate a function

dependence on the Lagrangian variables if it will not

generate ambiguity and misunderstanding.

3. Main Results

Given $\bar{\omega}(\bar{x}, t) \neq \bar{0}$, we define a vorticity direction field by the formula

$$\bar{\xi}(\bar{x}, t) \stackrel{\text{def}}{=} \frac{\bar{\omega}(\bar{x}, t)}{|\bar{\omega}(\bar{x}, t)|}, \quad \text{where } |\bar{\omega}(\bar{x}, t)| \stackrel{\text{def}}{=} \sqrt{\sum_{k=1}^3 \omega_k^2(\bar{x}, t)}. \text{ Let}$$

$$\Omega \stackrel{\text{def}}{=} \{(\bar{x}, t) \in (\mathbb{R}^3, \mathbb{R}_+ \cup \{0\}) : \bar{\omega}(\bar{x}, t) \neq \bar{0}\}. \text{ For } (\bar{x}, t) \in \Omega,$$

we define then the following scalar fields $\alpha(\bar{x}, t) \stackrel{\text{def}}{=} S \bar{\xi} \cdot \bar{\xi}$,

$$\zeta(\bar{x}, t) \stackrel{\text{def}}{=} \sqrt{S \bar{\xi} \cdot S \bar{\xi}}, \quad \beta(\bar{x}, t) \stackrel{\text{def}}{=} P \bar{\xi} \cdot \bar{\xi},$$

$$\gamma(\bar{x}, t) \stackrel{\text{def}}{=} P \bar{\xi} \cdot S \bar{\xi}, \quad \phi(\bar{x}, t) \stackrel{\text{def}}{=} \angle(S \bar{\xi}, \bar{\xi}),$$

where by $\angle(S \bar{\xi}, \bar{\xi})$ is designated the angle between $S \bar{\xi}$ and $\bar{\xi}$. Notice that

$\cos \phi = \alpha / \zeta$ is well-defined provided that

$S\bar{\xi} \neq \vec{0}$, because $\alpha = |S\bar{\xi}| |\bar{\xi}| \cos \phi = \zeta \cos \phi$. Since $\sec \phi = \frac{1}{\cos \phi}$, we see $\alpha \sec \phi = \zeta$.

Lemma 1. $\gamma = \alpha\beta - \left\{ \bar{\xi} \cdot \left[P\bar{\xi} \times (S\bar{\xi} \times \bar{\xi}) \right] \right\}$.

Proof. This formula is a corollary of that for the two-fold vector product. Indeed, according to the two-fold vector product formula

$\left[P\bar{\xi} \times (S\bar{\xi} \times \bar{\xi}) \right] = (P\bar{\xi} \cdot \bar{\xi})S\bar{\xi} - (P\bar{\xi} \cdot S\bar{\xi})\bar{\xi}$. Taking a scalar product of both the given equality sides with $\bar{\xi}$ we find that

$$\left\{ \left[P\bar{\xi} \times (S\bar{\xi} \times \bar{\xi}) \right] \cdot \bar{\xi} \right\} = (P\bar{\xi} \cdot \bar{\xi})(S\bar{\xi} \cdot \bar{\xi}) - (P\bar{\xi} \cdot S\bar{\xi})(\bar{\xi} \cdot \bar{\xi}) = \alpha\beta - \gamma$$

Let us derive differential equations for α and ζ . To do this let us compute the material derivatives of $S\bar{\omega} \cdot \bar{\omega}$ and

$S\bar{\omega} \cdot S\bar{\omega}$. Keeping in mind that $\frac{D\bar{\omega}}{Dt} = S\bar{\omega}$, $\frac{D(S\bar{\omega})}{Dt} = -P\bar{\omega}$,

$\frac{D|\bar{\omega}|}{Dt} = \alpha|\bar{\omega}|$, and $\alpha|\bar{\omega}| = |S\bar{\omega}| \cos \phi$, we obtain, on one hand, that

$$\begin{aligned} \frac{D(S\bar{\omega} \cdot \bar{\omega})}{Dt} &= \frac{D(S\bar{\omega})}{Dt} \cdot \bar{\omega} + S\bar{\omega} \cdot \frac{D\bar{\omega}}{Dt} \\ &= S\bar{\omega} \cdot S\bar{\omega} - P\bar{\omega} \cdot \bar{\omega} = (\zeta^2 - \beta)|\bar{\omega}|^2, \end{aligned}$$

and, on the other hand, that

$$\begin{aligned} \frac{D(S\bar{\omega} \cdot \bar{\omega})}{Dt} &= \frac{D(\alpha|\bar{\omega}|^2)}{Dt} = \\ |\bar{\omega}|^2 \frac{D\alpha}{Dt} + 2\alpha^2|\bar{\omega}|^2 &= \left[\frac{D\alpha}{Dt} + 2\alpha^2 \right] |\bar{\omega}|^2. \end{aligned}$$

Hence

$$\frac{1}{2} \frac{D(\alpha^2)}{Dt} = \alpha\zeta^2 - 2\alpha^2 - \alpha\beta. \tag{3}$$

Similarly, we find, on one hand, that

$$\frac{D(S\bar{\omega} \cdot S\bar{\omega})}{Dt} = 2S\bar{\omega} \cdot \frac{D(S\bar{\omega})}{Dt} = -2S\bar{\omega} \cdot P\bar{\omega} = -2\gamma|\bar{\omega}|^2, \text{ and,}$$

on the other hand, that

$$\begin{aligned} \frac{D(|S\bar{\omega}|^2)}{Dt} &= 2|S\bar{\omega}| \frac{D(|S\bar{\omega}|)}{Dt} = 2\zeta|\bar{\omega}| \frac{D(\zeta|\bar{\omega}|)}{Dt} \\ &= 2\zeta|\bar{\omega}| \left[|\bar{\omega}| \frac{D\zeta}{Dt} + \zeta \frac{D|\bar{\omega}|}{Dt} \right] \\ &= 2\zeta \left[\frac{D\zeta}{Dt} + \alpha\zeta \right] |\bar{\omega}|^2. \end{aligned}$$

Hence

$$\frac{1}{2} \frac{D(\zeta^2)}{Dt} = -\alpha\zeta^2 - \gamma. \tag{4}$$

Substituting $\alpha\beta - \left\{ \bar{\xi} \cdot \left[P\bar{\xi} \times (S\bar{\xi} \times \bar{\xi}) \right] \right\}$ instead of γ in Eq. (4) and subtracting Eq.(3) from the obtained equality term-wise we get finally the following differential equation

$$\begin{aligned} \frac{1}{2} \frac{D(\zeta^2 - \alpha^2)}{Dt} &= \\ -2\alpha(\zeta^2 - \alpha^2) + \left\{ \left[P\bar{\xi} \times (S\bar{\xi} \times \bar{\xi}) \right] \cdot \bar{\xi} \right\}. \end{aligned} \tag{5}$$

Definition. Two non-zero vectors of $\bar{\vartheta}$ and $\bar{\Theta}$ are said to be aligned perfectly if $\bar{\vartheta} \times \bar{\Theta} = \vec{0}$, i.e. when their vector (external) product is equal to zero.

Considering Eq.(5) one can easily see that $\zeta = \pm\alpha$ can be a solution to Eq.(5). Keeping in mind $\zeta^2 = \alpha^2 \sec^2 \phi$ one can easily find the differential equation for $\sec^2 \phi$:

$$\begin{aligned} \frac{1}{2} \frac{D \sec^2 \phi}{Dt} &= \left(\frac{\beta}{\alpha} - \alpha \sec^2 \phi \right) (\sec^2 \phi - 1) \\ &+ \frac{1}{\alpha^2} \left\{ \left[P\bar{\xi} \times (S\bar{\xi} \times \bar{\xi}) \right] \cdot \bar{\xi} \right\}. \end{aligned} \tag{6}$$

Considering Eq.(6) one can easily see that $\sec \phi = \pm 1$ can be solutions to Eq.(6). In particular, this is the case when $S\bar{\xi}$ and $\bar{\xi}$ are aligned perfectly. Indeed, by definition, the non-zero vectors $S\bar{\xi}$ and $\bar{\xi}$ are aligned perfectly provided that $S\bar{\xi} \times \bar{\xi} = \vec{0}$, i.e. when $\sec \phi = \pm 1$.

To define a notion of the restricted Euler equations, let us compute all

partial derivatives of the first equation of Eq.(1). In doing so we obtain the following equality

$$\frac{DU}{Dt} + U^2 = -P. \tag{7}$$

Taking the symmetrical and antisymmetrical parts of Eq.(3) we find that

$$\frac{DS}{Dt} + S^2 + A^2 = -P, \frac{DA}{Dt} + SA + AS = 0. \tag{8}$$

In general the Hessian P is the non-local non-isotropic matrix. Computing

div of Eqs. (7, 8) we find that $\Delta p = \text{tr}(P) = -\text{tr}(S^2 + A^2)$, where $\text{tr}(M)$ is a trace of the matrix M.

Definition. If to make violence and to assume P to be local and isotropic of the kind $P = -\frac{1}{3} \text{tr}(U^2)E$, we obtain then a model which is now called the "restricted Euler equations" [8].

Here, the elliptic pressure constraint, $\Delta p = \frac{\bar{\omega}^2}{2} - \text{tr}(S^2)$, is concerned solely with the diagonal elements of P.

Let us study the restricted Euler equations. Since, for any $\bar{x} \in R^3$ and $t \geq 0$, $P \bar{\xi} \stackrel{\text{def}}{=} -\frac{1}{3} \text{tr}(U^2)E \bar{\xi} = -\frac{1}{3} \text{tr}(U^2) \bar{\xi}$, $\bar{\xi}$ is the eigenvector of P.

The latter implies in turn that $\left\{ \left[P \bar{\xi} \times (S \bar{\xi} \times \bar{\xi}) \right] \cdot \bar{\xi} \right\} = 0$ identically, for all $\bar{x} \in R^3$ and $t \geq 0$.¹

This leads to considerable simplification of Eqs.(5,6) which take the following form:

$$\begin{aligned} \frac{D(\zeta^2 - \alpha^2)}{Dt} &= -4\alpha(\zeta^2 - \alpha^2), \\ \frac{D(\sec^2 \phi - 1)}{Dt} &= 2 \left(\frac{\beta}{\alpha} - \alpha \sec^2 \phi \right) (\sec^2 \phi - 1). \end{aligned} \tag{9}$$

If to use the Lagrangian variable, we can then rewrite Eq. (9) legitimately as

$$\begin{aligned} \frac{d \ln(\zeta^2 - \alpha^2)}{dt} &= -4\alpha, \\ \frac{d \ln(\sec^2 \phi - 1)}{dt} &= 2 \left(\frac{\beta}{\alpha} - \alpha \sec^2 \phi \right). \end{aligned} \tag{10}$$

Integrating Eq. (10) we obtain two families of solutions of the kind

$$\begin{aligned} (\zeta^2 - \alpha^2)|_{(\bar{X}(\bar{a}, t), t)} &= \\ \left[\zeta_o^2(\bar{a}) - \alpha_o^2(\bar{a}) \right] \exp \left\{ -4 \int_o^t \alpha(\bar{X}(\bar{a}, \tau), \tau) d\tau \right\}, \end{aligned} \tag{11}$$

$$\begin{aligned} (\sec^2 \phi - 1)|_{(\bar{X}(\bar{a}, t), t)} &= \\ \left[\sec_o^2(\bar{a}) - 1 \right] \exp \left\{ 2 \int_o^t \left[\frac{\beta(\bar{X}(\bar{a}, \tau), \tau)}{\alpha(\bar{X}(\bar{a}, \tau), \tau)} - \alpha(\bar{X}(\bar{a}, \tau), \tau) \sec^2 \phi(\bar{X}(\bar{a}, \tau), \tau) \right] d\tau \right\}, \end{aligned}$$

where $\zeta_o(\bar{a}) \stackrel{\text{def}}{=} \zeta_o(\bar{X}(\bar{a}, 0), 0)$, $\alpha_o(\bar{a}) \stackrel{\text{def}}{=} \alpha_o(\bar{X}(\bar{a}, 0), 0)$, $\phi_o(\bar{a}) \stackrel{\text{def}}{=} \phi_o(\bar{X}(\bar{a}, 0), 0)$. It is evident from Eq.(11) that $\zeta^2 = \alpha^2$ for $t > 0$ provided $\zeta_o^2(\bar{a}) = \alpha_o^2(\bar{a})$ as well as $\sec^2 \phi = 1$

for $t > 0$ provided $\sec^2 \phi_o(\bar{a}) = 1$ too. Thus we proved the following.

Proposition 1. Given the restricted Euler equations. If there is $\bar{a} \in R^3$ such that $\bar{\xi}_o(\bar{a}) \neq \bar{0} \neq S_o(\bar{a}) \bar{\xi}_o(\bar{a})$ are aligned perfectly, then $\bar{\xi}, S \bar{\xi}$ are aligned perfectly forever, for all admissible $t > 0$.

Let $\bar{e}_k, k=1,2,3$, be the eigenvectors of S associated with its eigenvalues $\lambda_k, k=1,2,3$. There is a generally accepted convention respecting ordering of λ_k : $\lambda_1 \geq \lambda_2 \geq \lambda_3$. It is not assumed in what follows. It is not difficult to show that if $\bar{\xi}$ is an eigenvector for S i.e., for example, $\bar{\xi} = \bar{e}_1$, then, in the frame of $\bar{e}_k, k=1,2,3$, the operators of A and A² take the following looks:

$$A = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{pmatrix}, A^2 = -\frac{1}{4} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & \omega^2 & 0 \end{pmatrix}.$$

In order to prove what is just aforesaid let us find now what is a matrix form of U, U^T, S, A in the orthonormal frame of eigenvectors of S provided that $\bar{\omega}$ is one of the eigenvectors of S.

Lemma 2. Given $\bar{\omega}$ be an eigenvector for S associated with its eigenvalue λ_ω . Then $\bar{\omega}, \lambda_\omega$ are the very same for U, U^T.

Lemma 3. Let $\bar{\omega}$ be an eigenvector for U or U^T. Then $\bar{\omega}$ is the same for S.

Proof of the lemmas. First of all notice that $A \bar{\omega} = \bar{0}$. Therefore, due to the definition of U, U^T, S, A, if $\bar{\omega}$ is the

¹ Indeed, taking into account that $\left[\bar{\xi} \times [S \bar{\xi} \times \bar{\xi}] \right] = S \bar{\xi} (\bar{\xi} \cdot \bar{\xi}) - (S \bar{\xi} \cdot \bar{\xi}) \bar{\xi}$ according to the two-fold vector product formula, what is the aforesaid can be reformulated with aid of the Lagrangian variables as follows: for all $\bar{x} \in R^3$ and $t \geq 0$, $\left\{ \bar{\xi}(\bar{X}(\bar{a}, t), t) \cdot \left[P(\bar{X}(\bar{a}, t), t) \bar{\xi}(\bar{X}(\bar{a}, t), t) \times (S(\bar{X}(\bar{a}, t), t) \bar{\xi}(\bar{X}(\bar{a}, t), t) \times \bar{\xi}(\bar{X}(\bar{a}, t), t)) \right] \right\} = -\frac{1}{3} \text{tr}(U^2(\bar{X}(\bar{a}, t), t)) \left\{ \bar{\xi}(\bar{X}(\bar{a}, t), t) \cdot \left[S(\bar{X}(\bar{a}, t), t) \bar{\xi}(\bar{X}(\bar{a}, t), t) \times \bar{\xi}(\bar{X}(\bar{a}, t), t) \right] \right\} = 0$.

eigenvector for S associated with its eigenvalue λ_ω , the one is the eigenvector for U , U^T associated with the very same eigenvalue and vice versa.

Remark. Keeping in mind that the eigenvalues of S all are real we can see that if $\bar{\omega}$ is the eigenvector for U or U^T , then one is associated to the mutual real eigenvalue of U and U^T .

It can be easily to check that there is a rotation symmetry of solutions to the restricted Euler equations in the sense of if $\bar{u}(\bar{x}, t), p(\bar{x}, t)$ is a solution pair to the restricted Euler equations, then $\bar{u}_Q(\bar{x}, t) \stackrel{\text{def}}{=} Q^T \bar{u}(Q\bar{x}, t), p_Q(\bar{x}, t) \stackrel{\text{def}}{=} p(Q\bar{x}, t)$ is the same pair for any real orthogonal matrix Q (i.e. such that $Q^T = Q^{-1}$). It is well-known also (see, e.g., [9], p. 258) that any real symmetric matrix is orthogonally similar to the diagonal one. Therefore, without loss of generality, we can assume that the aforementioned coordinate transformation is made from the very beginning and the Cartesian coordinates are chosen in such a manner that they coincides accurate to translation on some constant vector with the eigenvector frame of S and in doing so the vorticity direction $\bar{\xi}$ coincides, for instance, with that of the x_1 -axis, i.e. that $\bar{\omega} = \{\omega, 0, 0\}$. It is evident that, in the given frame, S is the diagonal matrix whose non-zero entries are $S_{j,j}, j = 1, 2, 3$. Keeping in mind this and that $\bar{\omega} = (U_{3,2} - U_{2,3}, U_{1,3} - U_{3,1}, U_{2,1} - U_{1,2})$ one can specify better entries of U in such a manner: $U_{2,1} = U_{1,2}, U_{3,1} = U_{1,3}, U_{3,2} = U_{2,3} + \omega$. Since $\bar{\xi}$ is the eigenvector for U and U^T , we find, as a corollary, that $U_{2,1} = U_{1,2} = 0, U_{3,1} = U_{1,3} = 0$, and also that $A_{2,1} = A_{1,2} = 0, A_{3,1} = A_{1,3} = 0$. Therefore, in the frame under consideration, the matrices U, U^T, S, A should have in general the following look:

$$U = \begin{pmatrix} U_{1,1} & 0 & 0 \\ 0 & U_{2,2} & U_{2,3} \\ 0 & U_{3,2} & U_{3,3} \end{pmatrix}, U^T = \begin{pmatrix} U_{1,1} & 0 & 0 \\ 0 & U_{2,2} & U_{3,2} \\ 0 & U_{2,3} & U_{3,3} \end{pmatrix},$$

$$S = \begin{pmatrix} S_{1,1} & 0 & 0 \\ 0 & S_{2,2} & 0 \\ 0 & 0 & S_{3,3} \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{2,3} \\ 0 & A_{3,2} & 0 \end{pmatrix}.$$

Remark. We notice attention that although the matrices U and U^T have a canonical look in the eigenvector frame of S , the ones are not normal. It can be easily checked if to

compute products UU^T, U^TU and to compare them then.

Applying to λ_k methods used in [5] for studying of behavior of eigenvalues of U one can show by the very same manner as it is done in [5] that, in the case $\bar{\xi} = \bar{e}_1$, dynamics of $\lambda_k, k = 1, 2, 3$ are described by the next equations:

$$\begin{cases} \frac{d\lambda_1}{dt} + \lambda_1^2 &= \frac{1}{3} \left(\sum_{j=1}^3 \lambda_j^2 - \frac{1}{2} |\omega|^2 \right), \\ \frac{d\lambda_2}{dt} + \lambda_2^2 - \frac{|\omega|^2}{4} &= \frac{1}{3} \left(\sum_{j=1}^3 \lambda_j^2 - \frac{1}{2} |\omega|^2 \right), \\ \frac{d\lambda_3}{dt} + \lambda_3^2 - \frac{|\omega|^2}{4} &= \frac{1}{3} \left(\sum_{j=1}^3 \lambda_j^2 - \frac{1}{2} |\omega|^2 \right). \end{cases} \quad (12)$$

Really, keeping in mind that $P = -\frac{1}{3} \text{tr}(U^2)E$, we can rewrite the first equation of Eq.(8) in such a manner:

$$\frac{\partial S}{\partial t} + (\nabla \cdot \bar{u})S + S^2 + A^2 = \frac{1}{3} \text{tr}(U^2)E. \quad (13)$$

Remark. It is worth to give attention that Eq.(12) is distinctly distinguished from that presented in [5,7] (see, for example, Eq.(6.2)[5] or Eq.(1.6)[7]). This difference occurs because of different objects under study: we consider the strain matrix S while the authors of [5-7] examine the velocity gradient matrix U .

Proposition 2. Given the restricted Euler equations. If $\bar{\xi} = \bar{e}_1$, i.e. is the eigenvector for S associated with its eigenvalue λ_1 , then the dynamics of $\lambda_k, k = 1, 2, 3$ is governed by Eq.(12).

Proof. $S\bar{e}_k = \lambda_k \bar{e}_k$ by definition. Differentiating this relation with respect to t we find $\frac{\partial S}{\partial t} \bar{e}_k + S \frac{\partial \bar{e}_k}{\partial t} = \frac{\partial \lambda_k}{\partial t} \bar{e}_k + \lambda_k \frac{\partial \bar{e}_k}{\partial t}$.

Computing a scalar product of the given equality terms with \bar{e}_k we obtain $\frac{\partial S}{\partial t} \bar{e}_k \cdot \bar{e}_k = \frac{\partial \lambda_k}{\partial t}$. Differentiation of the same relation with respect to x_j leads to $\frac{\partial S}{\partial x_j} \bar{e}_k + S \frac{\partial \bar{e}_k}{\partial x_j} = \frac{\partial \lambda_k}{\partial x_j} \bar{e}_k + \lambda_k \frac{\partial \bar{e}_k}{\partial x_j}$.

Multiplying this equality by u_j first and taking then a scalar product with \bar{e}_k we have $\left(u_j \frac{\partial S}{\partial x_j} \bar{e}_k \cdot \bar{e}_k \right) = u_j \frac{\partial \lambda_k}{\partial x_j}$.

Therefore, $\{[(\nabla \cdot \bar{u})S] \bar{e}_k \cdot \bar{e}_k\} = (\nabla \cdot \bar{u}) \lambda_k$.

Observing that $(S^2 \bar{e}_k \cdot \bar{e}_k) = \lambda_k^2, k = 1, 2, 3$ and that $(A^2 \bar{e}_1 \cdot \bar{e}_1) = 0$ as well as $(A^2 \bar{e}_k \cdot \bar{e}_k) = -\frac{1}{4} \omega^2, k = 2, 3$ and combining these facts with ones stated above one can easily see due to Eq.(13) that the proposition statement is valid in fact.

Subtracting the last equation of Eq.(12) from the previous one termwise we find after easy transformations that

$$\frac{d \ln(\lambda_2 - \lambda_3)}{dt} = -(\lambda_2 + \lambda_3) = \lambda_1.$$

Integrating this equation we obtain

$$\lambda_2 - \lambda_3 = [\lambda_2(0) - \lambda_3(0)] \exp \left\{ \int_0^t \lambda_1(\bar{X}(\bar{a}, \tau), \tau) d\tau \right\}. \quad (14)$$

On the other hand, recalling that $\frac{D\bar{\omega}}{Dt} = S\bar{\omega}$ and taking into account that $\bar{\omega}$ is the eigenvector for S associated with its eigenvalue λ_1 we see that $\frac{D|\bar{\omega}|}{Dt} = \lambda_1 |\bar{\omega}|$. Using the

Lagrangian variables converts it to the following $\frac{d|\bar{\omega}|}{dt} = \lambda_1 |\bar{\omega}|$. Integrating of the latter gives

$$|\bar{\omega}| = |\bar{\omega}_0(\bar{a})| \exp \left\{ \int_0^t \lambda_1(\bar{X}(\bar{a}, \tau), \tau) d\tau \right\}. \quad (15)$$

Comparing Eqs.(14-15) one can see that

$$\frac{|\bar{\omega}|}{|\bar{\omega}_0(\bar{a})|} = \frac{\lambda_2 - \lambda_3}{\lambda_2(0) - \lambda_3(0)}. \quad (16)$$

Thus, in the case under consideration, dynamics of $|\bar{\omega}|$ is the very same as that of difference of the matrix S eigenvalues which are not associated with $\bar{\omega}$ Substituting $-(\lambda_1 + \lambda_2)$ instead of λ_3 in Eq.(16) we find that

$$|\bar{\omega}|^2 = \frac{|\bar{\omega}_0(\bar{a})|^2 (\lambda_1 + 2\lambda_2)^2}{[\lambda_1(0) + 2\lambda_2(0)]^2} \stackrel{\text{def}}{=} a^2 (\lambda_1 + 2\lambda_2)^2.$$

If to substitute the expressions of λ_3 and $|\bar{\omega}|^2$ involving λ_1 and λ_2 in the first two equations of Eq.(12) only we obtain the next ones:

$$\begin{cases} \frac{d\lambda_1}{dt} = -\frac{\lambda_1^2}{2} + \frac{1}{6}(1-a^2)(\lambda_1 + 2\lambda_2)^2, \\ \frac{d\lambda_2}{dt} = -\frac{\lambda_2^2}{2} + \frac{1}{6}(2\lambda_1 + \lambda_2)^2 + \frac{a^2}{12}(\lambda_1 + 2\lambda_2)^2. \end{cases} \quad (17)$$

When $a^2 = 1$ we can easily integrate the first equation of Eq.(17).

$$\lambda_1 = \frac{2\lambda_1(0)}{2 + \lambda_1(0)t}. \quad (18)$$

We see that λ_1 does not blow up for a finite time provided $\lambda_1(0) \geq 0$ and blows up at $T = -2/\lambda_1(0)$ for any $\lambda_1(0) < 0$. In doing so $\lambda_1 \rightarrow -\infty$ as

$t \rightarrow T$. But what is interesting is $|\bar{\omega}| \rightarrow |\bar{\omega}_{T-o}(\bar{a})| < \infty$ as $t \rightarrow T$.

Unfortunately, we can say nothing respecting whether such the behavior of $\bar{\omega}$ contradicts to the celebrated BKM criterion [10] or not because the latter involves norms of \bar{u} in the Sobolev space $H^m(\mathbb{R}^3)$ and integrals with respect to time of those of $\bar{\omega}$ in the $L^\infty(\mathbb{R}^3)$ space. Let us pay attention also to that $|\lambda_k|, k = 2, 3$ enlarges at infinity as $t \rightarrow T$ although the difference $\lambda_2 - \lambda_3$ remains bounded up to time T. It is a corollary of $\sum_{k=1}^3 \lambda_k = 0$ and the behavior of λ_1 stated above.

Let us change a variable setting $\Lambda \stackrel{\text{def}}{=} \lambda_1 + 2\lambda_2$. In doing so Eq.(17) takes the look

$$\begin{cases} \frac{d\lambda_1}{dt} = -\frac{\lambda_1^2}{2} + \frac{1}{6}(1-a^2)\Lambda^2, \\ \frac{d\Lambda}{dt} = \lambda_1 \Lambda. \end{cases} \quad (19)$$

Multiplying the former of Eq.(19) on the latter crosswise we obtain $\lambda_1 \Lambda d\lambda_1 = -\frac{\lambda_1^2}{2} d\Lambda + \frac{1}{6}(1-a^2)\Lambda^2 d\Lambda$ or, otherwise,

$$d(9\lambda_1^2 \Lambda - [1-a^2]\Lambda^3) = 0.$$

Therefore, there is a first integral

$$9\lambda_1^2 \Lambda - [1-a^2]\Lambda^3 = b, \quad (20)$$

where

$$b = [\lambda_1(0) + 2\lambda_2(0)] \{ |\bar{\omega}_0(\bar{a})|^2 + 9\lambda_1^2(0) - [\lambda_1(0) + 2\lambda_2(0)]^2 \}.$$

We take into account here that $\Lambda(0) = \lambda_1(0) + 2\lambda_2(0)$ and

$$a^2 = \frac{|\bar{\omega}_0(\bar{a})|^2}{[\lambda_1(0) + 2\lambda_2(0)]^2}.$$

$$\frac{1}{6}(1-a^2)\Lambda^2 = \frac{3}{2}\lambda_1^2 - \frac{b}{6\Lambda}.$$

Solving Eq.(20) we find

Substituting the given expression in Eq.(19) we obtain

$$\begin{cases} \frac{d\lambda_1}{dt} = \lambda_1^2 - \frac{b}{6\Lambda}, \\ \frac{d\Lambda}{dt} = \lambda_1 \Lambda. \end{cases} \quad (21)$$

When $b=0$ we can easily integrate the first equation of Eq.(21).

Namely: $\lambda_1 = \frac{\lambda_1(0)}{1 - \lambda_1(0)t}$. We see that λ_1 has no blow up

for a finite time provided $\lambda_1(0) \leq 0$ and blows up at $T = 1/\lambda_1(0)$ for any $\lambda_1(0) > 0$. The same takes place for $\bar{\omega}$ and $\lambda_2 - \lambda_3$ too.

What is the aforesaid can be summarized in view of the following.

Proposition 3. Given the restricted Euler equations. If there is $\bar{a} \in \mathbb{R}^3$ such that $\bar{\xi}_o(\bar{a}) \neq \bar{0} \neq S_o(\bar{a}) \bar{\xi}_o(\bar{a})$ are aligned perfectly, then, for certain initial conditions, the eigenvalues of S lose their initial smoothness for a finite time by means of the so-called "finite-time blow up". As for the vorticity, it can as undergo the similar blow up at the same time as cannot.

Thus, an approach based on studying of the perfect alignment between the vorticity vector and the strain matrix eigenvectors permits us to deepen some results in [5] and to specify scenarios of the finite time blow ups of the restricted Euler equations solutions. To be more precise, Lemmas 6.1 - 6.2 [5] inform that the aforesaid solutions blow up at a finite time for almost all initial gradient matrix $U_o = U(0)$. But they say nothing how it occurs and what happens at the blow up time. The principal results presented above describe two concrete very different scenarios of the aforesaid blow ups.

Remark. We would like to attract a reader attention once more to that a

situation when the eigenvalues of S blow up for a finite time and in doing so the vorticity does not is, generally speaking, unexpected if to take into account the celebrated BKM criterion for solutions to the (genuine!) Euler equations to blow up at a finite time.

4. Conclusions

Summarizing all what is the aforesaid respecting the blow ups of solutions to the restricted Euler equations we can infer

that their properties differ from those of solutions to the genuine Euler equations very essentially. Really, as is just marked above there are parameter data and initial conditions such that the eigenvalues of the strain matrix S for the restricted Euler equations blow up for a finite time and in doing so their vorticity does not and vice versa. However, such kind a behavior is completely not possible for the Euler equations because, as it was shown by G.Ponce in [12], a finite-time blow up of the Euler equations vorticity $\bar{\omega}$ accompanies the very same one of the strain matrix S and, as a corollary, of its eigenvalues and vice versa. Therefore, the model under study cannot be considered as suitable one for recognizing and study of properties inherent to the Euler equations solutions. Nevertheless, in itself it is of considerable interest because exhibits unexpected features.

References

- [1] Viellefosse P.: Local interaction between vorticity and shear in a perfect incompressible fluid. J. Phys. Paris. vol.43 (1982), 837-842.
- [2] Viellefosse P.: Internal motion of a small element of fluid in an inviscid flow. Physica A. vol.125 (1984), 150-162.
- [3] Novikov E. A.: Internal dynamics of flows and formation of singularities. Fluid Dynam. Res. vol.6 (1990), 79-89.
- [4] Cantwell B. J.: Exact solution of a restricted Euler equation for the velocity gradient tensor. Phys. Fluids. vol.4, no. 4 (1992), 782-793.
- [5] Liu H. and Tadmor E.: Spectral dynamics of the velocity gradient field in restricted flows. Commun. Math. Phys. 228 (2002), 435-466.
- [6] Liu H. and Tadmor E.: Critical thresholds in 2D restricted Euler-Poisson equations. SIAM J. Appl. Math., vol. 63, no. 6 (2003), 1889-1910.
- [7] Liu H., Tadmor E. and Wei D.: Global regularity of the 4D restricted Euler equations. Physica D (2009), doi: 10.1016/j.phesd.2009.07.009.
- [8] Gibbon J. D.: The three-dimensional Euler equations: Where do we stand? Physica D. vol. 237 (2008), 1894-1904.
- [9] Gantmakher F. R., Theory of Matrices, 3th edition, "Nauka" Publishing House, Moscow, 1967, 575 p. (in Russian).
- [10] Beale J. T., Kato T. and Majda A.: Remarks on the breakdown of smooth solutions for 3-D Euler equations. Commun. Math. Phys. 94 (1984), 61-66.
- [11] Dobrynskii V. A.: Perfect vector alignment of the vorticity and the vortex stretching is one forever. Forum Mathematicum (2011, september), DOI~10.1515~FORM.2011.149, 13 pp.
- [12] Ponce G.: Remarks on a paper by J.T.Beale, T.Kato and A.Majda. Commun. Math. Phys. 98 (1985), 349-353.