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An New Iterative Scheme for Variational Inequalities and Nonexpansive Mappings in Hilbert Spaces

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Abstract

In this paper, a new three-step iterative scheme is introduced for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality for α-inverse-strongly monotone mappings. The result reveals that the proposed iterative sequence converges strongly to the common element of this two. And our studies can be regarded as an extension of the existing results, which we illustrate one by one in our remarks.

Keywords

Variational Inequality, Nonexpansive Mapping, Fixed Point

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1. Introduction

Variational inequalities first studied by Stampacchia [1] in 1960s have played an important role in the development of pure and applied mathematics. They have also witnessed an explosive growth in theoretical progression, algorithmic development, etc.; see e.g. [2-14]. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot , \cdot \rangle$ and $\|.\|$, respectively. Let C be a nonempty closed convex subset of H and A a mapping from C to H. The classical variational inequality problem is to find a vector $u \in \mathrm{C}$ such that

$$\langle \mathrm{A}u, v-u\rangle \geq 0,$$

For all or $v \in \mathrm{C}$. The set of solutions of the variational inequality is denoted by $\mathrm{VI(C,A)}$. A mapping A of C to H is called α-inverse-strongly monotone [6] if there exists a positive real number α such that

$$\langle \mathrm{A}u, v-u\rangle \geq \alpha\|\mathrm{A}u-\mathrm{A}v\|^2,$$

for any $u, v \in \mathrm{C}$. A mapping S of C into itself is called nonexpansive [6] if

$$\|Su-Sv\| \leq \|u-v\|,$$

for all $u, v \in \mathrm{C}$. We denote the set of fixed points of S by F(S).

In order to seek for an element of $\mathrm{F(S)} \cap \mathrm{VI(C,A)}$. Takahashi and Toyoda [4] introduced the following iterative scheme

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) SP_C(I-\tau_n A)x_n, \tag{1}$$

for every $n = 0,1,2,\ldots$, where $x_0 \in C, \{\alpha_n\}$ is a sequence in $(0,1)$ and τ_n is a sequence in $(0,2\alpha), P_C$ is the metric projection of H onto C. They proved that the iterative consequences $\{x_n\}$ generated by (1) converge weakly to an element $q \in F(S) \cap VI(C,A) \neq \emptyset$. For convenience, we will use $F = F(S) \cap VI(C,A) \neq \emptyset$ through the whole paper.

On the other hand, Iiduka and Takahashi [5] put forward another iterative scheme:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\, SP_C(I-\tau_n A)x_n, \tag{2}$$

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for every $n = 0,1,2,\ldots$, where $x_0 \in C, \{\alpha_n\}$ is a sequence in $(0,1)$ τ_n is a sequence in $(0,\ 2\alpha)$, and P_C is the metric projection of H onto C. It was proved that the iterative consequences $\{x_n\}$ generated by (2) converge strongly to an element $q \in F$.

Furthermore, Yao and Yao [6] proposed the following mixed gradient method:

$$\begin{cases} x_1 = u \epsilon C, \\ y_n = P_C(I - \tau_n A)x_n \\ x_{n+1} = \alpha_n u + \beta_n x_n + \vartheta_n SP_C(I - \tau_n A)y_n, \end{cases} \tag{3}$$

for every $n = 1,2,\ldots$, where $\{\alpha_n\}$, $\{\beta_n\}, \{\vartheta_n\}$ are sequences in $(0,1)$ satisfied $\alpha_n + \beta_n + \vartheta_n = 1$, and τ_n is a sequence in $(0, 2\alpha)$, P_C is the metric projection of H onto C. They proved that the iterative consequences defined by (3) converge strongly to $q = P_F u$, , where $P_F u$ was the metric projection of u onto F.

In recent years, many authors have studied some different iterative schemes both in Hilbert spaces and Banach spaces, see e.g. [2-14]. Inspired and motivated by those previous researches, we suggest and analyze a new iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the solution set of a variational inequality for α-inverse-strongly monotone mapping in real Hilbert spaces. Strong convergence theorems are established and the iterative methods considered by [4, 5, 7, 10] are included in our results.

2. Preliminary

For convenience, we would like to list some definitions and fundamental lemmas which are useful in the following consequent analysis. They can be found in any standard functional analysis books such as [15, 16].

Definition 2.1 A mapping $f: \mathrm{C} \to \mathrm{C}$ is a contraction on C if there exists a constant $k \in (0,1)$ such that

$$\|f(u) - f(v)\| \le k\|u - v\|, \forall\, u, v \in C.$$

Definition 2.2 A set-valued mapping $\mathrm{T}: \mathrm{H} \to 2^H$ is called monotone if for all $x, y \in \mathrm{H}, f \in \mathrm{T}x$ and $g \in \mathrm{T}y$ imply $\langle x - y, f - g\rangle \ge 0$.

Definition 2. 3 A monotone mapping $\mathrm{T}: \mathrm{H} \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H, \langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H and let $N_C(.)$ be the normal cone operator to C defined by $N_C(v) = \{w \in \mathrm{H}: \langle w, v - u\rangle \ge 0, \forall\, u \in C\}$. Define

$$T(v) = \begin{cases} A(v) + N_C(v), v \in C, \\ \emptyset, \quad otherwise. \end{cases} \tag{4}$$

Then T is maximal monotone and $0 \in T(v)$ if and only if $v \in VI(C, A)$ (see [11]).

Definition 2.4 For every point $x \in H$, there exists a unique nearest point u in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C.

It is well known that P_C is a nonexpansive mapping of H onto C and satisifies $\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2$ for every $x, y \in H$.

Moreover, P_C is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \text{ for all } x \in H, y \in C \tag{5}$$

It is easy to see that the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \tau A u), \forall\, \tau > 0. \tag{6}$$

Note that H satisfies Opial's condition [17], i.e., for any sequence $\{z^k\}$ with $z^k \to z_0$, the inequality

$\lim_{k\to\infty} inf\|z^k - z_0\| < \lim_{k\to\infty} inf\|z^k - z\|$ holds for every $z \in H$ with $z \ne z_0$.

Next we present some useful lemmas.

The following lemma is an immediate consequence of equality:

$$\|x + y\|^2 = \|x\|^2 + 2\langle y, x + y\rangle - \|y\|^2.$$

Lemma 2. 1 Let H be a real Hilbert space. Then the following inequality holds:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \forall\, x, y \in H.$$

Lemma 2. 2 (Osilike [14]) Let $(E, \langle ., .\rangle)$ be an inner space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2$$

$$-\alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2. 3 (Xu [10]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1 - \omega_n)\, a_n + \epsilon_n$, where ω_n is a sequence in $(0,1)$ and ϵ_n is a sequence such that

(i). $\sum_{n=0}^{n=\infty} \omega_n = \infty$;

(ii). $\lim_{n\to\infty} sup \frac{\epsilon_n}{\omega_n} \le 0$ or $\sum_{n=0}^{n=\infty} |\epsilon_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For convenience, we use → for strong convergence and ⇢ for weak convergence in the following analysis.

3. Main Results

In this section, we suggest and analyze a new iterative scheme for finding the common element of the fixed points of a nonexpansive mapping and the solution set of variational inequalities for an α-inverse-strongly monotone mapping in a real Hilbert space. Strong convergence theorems are established and several special cases are also discussed.

Theorem 3. 1 Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be an α -inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $\mathrm{F} = \mathrm{F(S)} \cap \mathrm{VI(C, A)} \neq \emptyset, f: \mathrm{C} \to \mathrm{C}$ be a contraction mapping with coefficient $k \in (0,1)$. Suppose $x_0 \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ are given by

$$\begin{cases} y_n = \theta_n x_n + (1 - \theta_n) SP_C(I - \tau_n A)x_n, \\ z_n = \beta_n f(x_n) + (1 - \beta_n) P_C(I - \tau_n A)y_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\, Sz_n, n \geq 0, \end{cases} \quad (7)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\theta_n\}$ are three sequences in [0,1] and $\{\tau_n\}$ is a sequence in $[0,2\alpha]$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\theta_n\}$ are chosen so that $\{\tau_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\alpha$, and

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{n=\infty} \alpha_n = \infty, \sum_{n=0}^{n=\infty} |\alpha_n - \alpha_{n-1}| < \infty$,

(C2) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=0}^{n=\infty} \beta_n = \infty, \sum_{n=0}^{n=\infty} |\beta_n - \beta_{n-1}| < \infty, \lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$,

(C3) $\sum_{n=0}^{n=\infty} |\tau_n - \tau_{n-1}| < \infty, \sum_{n=0}^{n=\infty} |\theta_n - \theta_{n-1}| < \infty, \lim_{n\to\infty} \theta_n = 1$.

Then the sequence $\{x_n\}$ converges strongly to $q \in \mathrm{F}$, where $q = P_F f(q)$ or equivalently q satisfies the following variational inequality:

$$\langle (I - f)q, q - p \rangle \leq 0, \forall\, p \in F.$$

Proof: We first show that $\mathrm{I} - \tau_n A$ is a nonexpansive mapping. For all $x, y \in H$, and $\tau_n \in [0,2\alpha]$, we have

$$\begin{aligned} \|(\mathrm{I} - \tau_n A)x - (\mathrm{I} - \tau_n A)y\|^2 &= \|x - y\|^2 - 2\tau_n \langle x - y, Ax - Ay \rangle \\ &\quad + {\tau_n}^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \tau_n(\tau_n - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|, \end{aligned}$$

which implies that $\mathrm{I} - \tau_n A$ is nonexpansive .

For convenience, we set $P_n = P_C(\mathrm{I} - \tau_n A)x_n, Q_n = P_C(\mathrm{I} - \tau_n A)y_n$.

Then the iterative scheme (7) can be written as:

$$\begin{cases} y_n = \theta_n x_n + (1 - \theta_n) SP_n, \\ z_n = \beta_n f(x_n) + (1 - \beta_n) Q_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\, Sz_n, n \geq 0. \end{cases} \quad (8)$$

Let $p \in \mathrm{F}$. Then we have $p = P_C(\mathrm{I} - \tau_n A)p$ by (6) and $Sp = p$. Since the proof of the theorem is rather long, it will be more convenient to divide the process into several steps.

Step 1. We claim that $\{x_n\}$ is bounded.

Since both P_C and $\mathrm{I} - \tau_n A$ are nonexpansive mappings, we have

$$\|P_n - p\| \leq \|(\mathrm{I} - \tau_n A)x_n - (\mathrm{I} - \tau_n A)p\| \leq \|x_n - p\|. \quad (9)$$

Similarly, we obtain that

$$\|Q_n - p\| \leq \|y_n - p\|. \quad (10)$$

Combining (8) and (9), together with that S is nonexpansive mapping, we see that

$$\|y_n - p\| \leq \|x_n - p\|. \quad (11)$$

By (10) and (11), we get

$$\|Q_n - p\| \leq \|x_n - p\|.$$

Hence

$$\|z_n - p\| \leq [1 - \beta_n(1 - k)]\|x_n - p\| + \beta_n \|f(p) - p\|. \quad (12)$$

From (12), we arrive at

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n)\|z_n - p\| \\ &\leq \max_{n\geq 0}\{\|x_n - p\|, \tfrac{\|f(p)-p\|}{1-k}\}. \end{aligned} \quad (13)$$

By the method of induction, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \tfrac{\|f(p)-p\|}{1-k}\} \triangleq M_1. \quad (14)$$

Therefore $\{x_n\}$ is bounded. Consequently, all those sequences $\{P_n\}, \{Q_n\}, \{SP_n\}, \{Ay_n\}, \{Ax_n\}, and \{f(x_n)\}$ are bounded.

Step 2. We now in the position to prove that $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$.

Since both P_C and $\mathrm{I} - \tau_n A$ are nonexpansive mappings, we first have

$$\|P_n - P_{n-1}\| \leq \|x_n - x_{n-1}\| + |\tau_n - \tau_{n-1}| \cdot \|Ax_{n-1}\|. \quad (15)$$

By similar method, we have

$$\|Q_n - Q_{n-1}\| \leq \|y_n - y_{n-1}\| + |\tau_n - \tau_{n-1}| \cdot \|Ay_{n-1}\|. \quad (16)$$

In view of (15), after simple calculation, we see that

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq \|x_n - x_{n-1}\| + |\tau_n - \tau_{n-1}| \cdot \|Ax_{n-1}\| \\ &\quad + |\theta_n - \theta_{n-1}|(\|x_{n-1}\| + \|SP_{n-1}\|). \end{aligned} \quad (17)$$

By (16) and (17), we get

$\|z_n - z_{n-1}\| \leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|Q_n\|)$

$+ |\theta_n - \theta_{n-1}|(\|x_{n-1}\| + \|SP_{n-1}\|)$

$+ |\tau_n - \tau_{n-1}|(\|Ax_{n-1}\| + \|Ay_{n-1}\|).$ (18)

In view of (18), we have

$$\|z_n - z_{n-1}\| \leq [1 - \alpha_n(1-k)]\|x_n - x_{n-1}\| + \mu_n, \quad (19)$$

Where

$\mu_n = |\alpha_n - \alpha_{n-1}|[\|f(x_{n-1})\| + \|Sz_{n-1}]\|] + |\beta_n - \beta_{n-1}|(\|f(x_{n-1})\| + \|Q_n\|)$

$+ |\theta_n - \theta_{n-1}|(\|x_{n-1}\| + \|SP_{n-1}\|)$

$+ |\tau_n - \tau_{n-1}|(\|Ax_{n-1}\| + \|Ay_{n-1}\|).$ (20)

By the conditions (C1), (C2) and (C3), we see that $\sum_{n=0}^{n=\infty}|\mu_n| < \infty$, and $\sum_{n=0}^{n=\infty}(1-k)\alpha_n = \infty$, which combining with Lemma 2.3, yields

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \quad (21)$$

Since $x_{n+1} - x_n = \alpha_n[f(x_n) - x_n] + (1-\alpha_n)(Sz_n - x_n)$, together with (21) and the condition (C1) imply that

$$\lim_{n\to\infty}\|Sz_n - x_n\| = 0. \quad (22)$$

Since $\|x_n - y_n\| = (1-\theta_n)\|x_n - SP_n\|$, $\lim_{n\to\infty}\theta_n = 1$, and $\|x_n - SP_n\|$ is bounded, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \quad (23)$$

Furthermore, combining Lemma 2.2 with that $\mathrm{I} - \tau_n A$ is nonexpansive, A is α-inverse-strongly monotone mapping, $\{\tau_n\} \subset [a,b]$, and $0 < a < b < 2\alpha$, we obtain that

$\|z_n - p\|^2 = \|\beta_n[f(x_n) - p] - (1-\beta_n)(Q_n - p)\|^2$

$\leq \beta_n\|f(x_n) - p\|^2$

$+(1-\beta_n)[\|x_n - p\|^2 + a(b-2\alpha)\|Ay_n - Ap\|^2].$ (24)

From (24), together with Lemma 2.2, we see that

$\|x_n - p\|^2 = \|\alpha_n[f(x_n) - p] + (1-\alpha_n(Sz_n - p)\|^2$

$\leq [\alpha_n + (1-\alpha_n)\beta_n]\|f(x_n) - p\|^2$

$+\|x_n - p\|^2 + a(b-2\alpha)\|Ay_n - Ap\|^2,$ (25)

which implies that

$-a(b-2\alpha)\|Ay_n - Ap\|^2 \leq [\alpha_n + (1-\alpha_n)\beta_n]\|f(x_n) - p\|^2$

$+\|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|).$ (26)

It follows from conditions (C1), (C2) and (21) that

$$\lim_{n\to\infty}\|Ay_n - p\| = 0. \quad (27)$$

Step 3. We show that $\lim_{n\to\infty}\|SQ_n - Q_n\| = 0$.

Since $\|SQ_n - Sz_n\| \leq \|Q_n - z_n\| \leq \beta_n\|f(x_n) - Q_n\|$, $\|f(x_n) - Q_n\|$ is bounded, and $\lim_{n\to\infty}\beta_n = 0$, we have

$$\lim_{n\to\infty}\|SQ_n - Sz_n\| = 0. \quad (28)$$

We now show that $\lim_{n\to\infty}\|Q_n - y_n\| = 0$.

Since

$\|Q_n - p\|^2 \leq \langle(\mathrm{I} - \tau_n A)y_n - (\mathrm{I} - \tau_n A)p, Q_n - p\rangle$

$= \frac{1}{2}\{\|(\mathrm{I} - \tau_n A)y_n - (\mathrm{I} - \tau_n A)p\|^2 + \|Q_n - p\|^2$

$-\|[(\mathrm{I} - \tau_n A)y_n - (\mathrm{I} - \tau_n A)p] - (Q_n - p)\|^2\}$

$\leq \frac{1}{2}[\|y_n - p\|^2 + \|Q_n - p\|^2 - \|y_n - Q_n\|^2$

$+2\tau_n\langle y_n - Q_n, Ay_n - Ap\rangle],$ (29)

we get that

$\|Q_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - Q_n\|^2 + 2\tau_n\|y_n - Q_n\| \cdot \|Ay_n - Ap\|.$ (30)

Hence

$\|x_{n+1} - p\|^2 \leq [\alpha_n + (1-\alpha_n)\beta_n]\|f(x_n) - p\|^2$

$+(1-\alpha_n)(1-\beta_n)\|Q_n - p\|^2.$ (31)

It follows from (30) that

$\|x_{n+1} - p\|^2 \leq [\alpha_n + (1-\alpha_n)\beta_n]\|f(x_n) - p\|^2 + \|x_n - p\|^2$

$-(1-\alpha_n)(1-\beta_n)\|y_n - Q_n\|^2$

$+2\tau_n\|y_n - Q_n\| \cdot \|Ay_n - Ap\|.$ (32)

Hence

$(1-\alpha_n)(1-\beta_n)\|y_n - Q_n\|^2 \leq [\alpha_n + (1-\alpha_n)\beta_n]\|f(x_n) - p\|^2$

$+\|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|)$

$+2\tau_n\|y_n - Q_n\| \cdot \|Ay_n - Ap\|.$ (33)

Since

$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|Ay_n - Ap\| = 0$, and $\|f(x_n) - p\|$, $\|x_n - p\| + \|x_{n+1} - p\|$, $\|y_n - Q_n\|$ are bounded,

we have

$$\lim_{n\to\infty}\|y_n - Q_n\| = 0. \quad (34)$$

It follows from (28), (22),(23) and (34) , together with

$$\|SQ_n - Q_n\| \leq \|SQ_n - Sz_n\| + \|Sz_n - x_n\| + \|x_n - y_n\| + \|y_n - Q_n\|$$

that

$$\lim_{n\to\infty}\|SQ_n - Q_n\| = 0. \quad (35)$$

Step 4. We prove that $q_0 \in F$.

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to q_0.

Since

$\|Q_n - x_n\| \le \|Q - y_n\| + \|y_n - x_n\|$, combining (23) and (34) we know that $\lim_{n\to\infty}\|Q_n - x_n\| = 0$.

Then $Q_{n_i} \rightharpoonup q_0$.

Next we show that $q_0 \in VI(C, A)$.

Let

$$T(v) = \begin{cases} A(v) + N_C(v), v \in C, \\ \emptyset, \ otherwise. \end{cases}$$

where $N_C(v) = \{w \in H: \langle v - u, w\rangle \ge 0, \forall u \in C\}$. Then T is maximal monotone. Let $(v, w) \in G(T)$, where $G(T) = \{(v, w): w \in Tv\}$.

Since $w - Av \in N_C v$ and $Q_n \in C$, we have $\langle v - Q_n, w - Av\rangle \ge 0, \forall n \ge 0$.

On the other hand, from (5) and $Q_n = P_C(I - \tau_n A)y_n$,

we see that $\langle v - Q_n, Q_n - (I - \tau_n A)y_n\rangle \ge 0, \forall n \ge 0$. Then

$$\langle v - Q_{n_i}, \frac{Q_{n_i} - y_{n_i}}{\tau_{n_i}} + Ay_{n_i}\rangle \ge 0, \forall n \ge 0.$$

Thus

$\langle v - Q_{n_i}, w\rangle \ge \langle v - Q_{n_i}, Av\rangle$

$\ge \langle v - Q_{n_i}, Av\rangle - \langle v - Q_{n_i}, \dfrac{Q_{n_i} - y_{n_i}}{\tau_{n_i}} + Ay_{n_i}\rangle$

$= \langle v - Q_{n_i}, Av - Ay_{n_i} - \dfrac{Q_{n_i} - y_{n_i}}{\tau_{n_i}}\rangle$

$$\ge \langle v - Q_{n_i}, AQ_{n_i} - Ay_{n_i}\rangle - \langle v - Q_{n_i}, \frac{Q_{n_i} - y_{n_i}}{\tau_{n_i}}\rangle. \quad (36)$$

Putting $i \to \infty$, we have $\langle v - q_0, w\rangle \ge 0$.

Since T is maximal, we have $q_0 \in T^{-1}(0)$. Hence $q_0 \in VI(C, A)$.

Now let us show that $q_0 \in F(S)$. Assume that $q_0 \notin F(S)$. From Opial's condition, we have

$\liminf_{i\to\infty} \|Q_{n_i} - q_0\| < \liminf_{i\to\infty} \|Q_{n_i} - Sq_0\|$

$\le \liminf_{i\to\infty} \|SQ_{n_i} - Sq_0\|$

$$\le \lim_{i\to\infty} inf \|Q_{n_i} - q_0\|. \quad (37)$$

This is a contradiction. Thus we obtain that $q_0 \in F(S)$.

Since $P_F f$ is a contraction mapping, by Banach's contraction theorem, there exists a unique fixed point q of $P_F f$, that's $q = P_F f(q)$.

Step 5. We prove that $\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle \le 0$.

From (5), we know

$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \limsup_{i\to\infty}\langle f(q) - q, Q_n - q\rangle$

$$= \limsup_{i\to\infty}\langle f(q) - q, Q_{n_i} - q\rangle = \langle f(q) - q, q_0 - q\rangle \le 0. \quad (38)$$

Step 6. We claim that $x_n \to q$. From Lemma 2.1 and Lemma 2.2, we obtain that

$\|x_{n+1} - q\|^2 = \|\alpha_n[f(x_n) - q] + (1 - \alpha_n)\{S[\beta_n[f(x_n) + (1 - \beta_n)Q_n]\} - Sq\|^2$

$\le (1 - \alpha_n)^2\|S[\beta_n[f(x_n) + (1 - \beta_n)Q_n - Sq]\|^2 + 2\alpha_n\langle f(x_n) - q, x_{n+1} - q\rangle$

$\le [(1 - \alpha_n)^2 + \alpha_n k]\|x_n - q\|^2 + 2\beta_n(1 - \alpha_n)^2\|f(x_n) - q\| \cdot \|f(x_n) - q\|$

$$+\alpha_n k\|x_{n+1} - q\|^2 + 2\alpha_n\langle (q) - q, x_{n+1} - q\rangle. \quad (39)$$

Then we have

$(1 - \alpha_n k)\|x_{n+1} - q\|^2 \le [1 - (2 - k)\alpha_n + \alpha_n{}^2]\|x_n - q\|^2$

$+2\beta_n(1 - \alpha_n)^2\|f(q) - q\| \cdot \|f(x_n) - q\|$

$$+2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle. \quad (40)$$

That is

$\|x_{n+1} - q\|^2 \le \dfrac{1 - (2 - k)\alpha_n + \alpha_n{}^2}{1 - \alpha_n k}\|x_n - q\|^2$

$+\dfrac{2\beta_n(1 - \alpha_n)^2}{1 - \alpha_n k}\|f(q) - q\| \cdot \|f(x_n) - q\|$

$+\dfrac{2\alpha_n}{1 - \alpha_n k}\langle f(q) - q, x_{n+1} - q\rangle$

$\le \left(1 - \dfrac{2(1 - k)\alpha_n}{1 - \alpha_n k}\right)\|x_n - q\|^2$

$$+\frac{2(1-k)\alpha_n}{1-\alpha_n k}\left[\frac{\beta_n(1-\alpha_n)^2}{(1-k)\alpha_n}M_2 + \frac{\alpha_n}{2(1-k)}M_3 + \frac{\langle f(q)-q, x_{n+1}-q\rangle}{1-k}\right], \quad (41)$$

where $M_2 = sup_{n\ge 0}\{\|f(q) - q\| \cdot \|f(x_n) - q\|\}$, and $M_3 = sup_{n\ge 0}\{\|x_n - q\|^2\}$.

From (38) and conditions (C1), (C2) and (C3), letting $n \to \infty$ yields

$$\lim_{n\to\infty} sup \left[\frac{\beta_n(1-\alpha_n)^2}{(1-k)\alpha_n} M_2 + \frac{\alpha_n}{2(1-k)} M_3 + \frac{\langle f(q)-q, x_{n+1}-q\rangle}{1-k}\right] \leq 0. \quad (42)$$

Let

$$s_n = \frac{2(1-k)\alpha_n}{1-\alpha_n k}, t_n = \frac{2(1-k)\alpha_n}{1-\alpha_n k}\left[\frac{\beta_n(1-\alpha_n)^2}{(1-k)\alpha_n} M_2 + \frac{\alpha_n}{2(1-k)} M_3 + \frac{\langle f(q)-q, x_{n+1}-q\rangle}{1-k}\right].$$

Then $\|x_{n+1}-q\|^2 \leq (1-s_n)\|x_n-q\|^2 + t_n$.

It is easy to check that $s_n \to 0, \sum_{n=0}^{n=\infty} s_n = \infty, \lim_{n\to\infty} sup \frac{t_n}{s_n} \leq 0$.

By Lemma 2.3, we see that

$$\lim_{n\to\infty}\|x_n - q\| = 0. \quad (43)$$

The proof is finished.

As an implication of Theorem 3.1, we have the following corollary:

Corollary 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be an α -inverse-strongly monotone mapping of C into H and let S be a be a nonexpansive mapping of C into itself such that $\mathrm{F} = \mathrm{F(S)} \cap \mathrm{VI(C, A)} \neq \emptyset, f : \mathrm{C} \to \mathrm{C}$ be a contraction mapping with coefficient k$\in (0,1)$. Suppose that $x_0 \in C$ and $\{x_n\}, \{y_n\}, \{z_n\}$ are given by

$$\begin{cases} y_n = \theta_n x_n + (1-\theta_n)P_C(I - \tau_n A)x_n, \\ z_n = \beta_n f(x_n) + (1-\beta_n)P_C(I - \tau_n A)y_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\, \mathrm{S} z_n, n \geq 0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\theta_n\}$ are three sequences in [0,1] and $\{\tau_n\}$ is a sequence in $[0, 2\alpha]$. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\theta_n\}$ are chosen so that $\{\tau_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\alpha$, and

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{n=\infty} \alpha_n = \infty, \sum_{n=0}^{n=\infty} |\alpha_n - \alpha_{n-1}| < \infty$,

(C2) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=0}^{n=\infty} \beta_n = \infty, \sum_{n=0}^{n=\infty} |\beta_n - \beta_{n-1}| < \infty, \lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$,

(C3) $\sum_{n=0}^{n=\infty} |\tau_n - \tau_{n-1}| < \infty, \sum_{n=0}^{n=\infty} |\theta_n - \theta_{n-1}| < \infty, \lim_{n\to\infty} \theta_n = 1$.

Then the sequence $\{x_n\}$ converges strongly to q ∈ F, where $\mathrm{q} = P_F f(q)$ or equivalently q satisfies the following variational inequality:

$$\langle (I-f)q, q-p\rangle \leq 0, \forall\, p \in F.$$

Proof: The conclusion follows from Theorem 3.1 by setting $S = I$.

Theorem 3.1 extends the corresponding results of [4, 5, 7, 10].

Remark 3.1 Putting $f = I, \beta_n = 1, \theta_n = 1$ in Theorem 3.1, we can get the iterative scheme provided by [4].

Remark 3.2 Putting $f(x_n) = x_0, \beta_n = 0, \theta_n = 1$ in Theorem 3.1, we can get the iterative scheme provided by [5].

Remark 3.3 The proposition 3.1 of [7] is a special case of our result.

In fact, letting $\theta_n = 1, \beta_n = 0$ in Theorem 3.1, we get

$$x_0 \in C, x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)SP_C(I - \tau_n A)x_n.$$

Then

$$x_n \to q \in F(S) \cap VI(C, A) \text{ by Theorem 3.1.}$$

Remark 3.4 Putting $\beta_n = 1, \theta_n = 0$ in Theorem 3.1, we can get the iterative scheme provided by [10].

Remark 3.5 The conditions in Theorem 3.1 can be easily satisfied, for example

$$\alpha_n = \frac{1}{\sqrt{n+8}}, \beta_n = \frac{1}{n+8}, \tau_n = \frac{1}{2n}, \theta_n = \frac{n-1}{n}.$$

4. Conclusion

By introducing a new iterative scheme for variational inequalities and nonexpansive mappings in Hilbert spaces, we proved that the sequences generated by the iterative scheme strongly converge to a common element of the fixed points of a nonexpansive mapping and the solution set of variational inequality for α -inverse-strongly monotone mapping.

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