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Power Series Solutions to Generalized Abel Integral Equations

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Abstract

Even though they have a rather specialized structure, Abel equations form an important class of integral equations in applications. This happens because completely independent problems lead to the solution of such equations. In this paper we consider the generalized Abel integral equation of the first and second kind. Authors have been proposed a new method for constructing solutions of Abel by a power series.

Keywords

Generalized Abel Integral Equations, Integral Equation, Power Series

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1. Introduction

The real world problems in scientific fields such as solid state physics, plasma physics, fluid mechanics, chemical kinetics and mathematical biology are nonlinear in general when formulated as partial differential equations or integral equations. Abel's integral equation occurs in many branches of scientific fields [1], [2] such as microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostics, X-ray radiography, and optical fiber evaluation. Abel's integral equation is the earliest example of an integral equation [5], [9]. In this paper, we use the method of generalized power series, to solve linear Volterra integral equations of the first and second kind. This power series are undetermined coefficients method, or a method based on the application of the Taylor series. The result obtained in the form of generalized power series solution further converted to the inversion formula of the integral equation. One such method is the representation of the solution of the equation in the form of a power series [8], [10]. Moreover, the basic theorems of this section are given without proof.

2. Main Results

Consider the generalized Abel integral equation of the first kind:

$$\int_a^x \frac{\phi(t)dt}{(x-t)^\alpha} = f(x), \tag{1}$$

where $0<\alpha<1$ – arbitrary real constant, $f(x):[a,b]\to R$ is given function.

Let's assume that the function $f(x)$ can be represented as follows:

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+...+c_n(x-a)^n+... \tag{2}$$

We shall seek for a solution equation in the form of the following generalized power series:

$$\phi(t)=a_0(t-a)^\gamma+a_1(t-a)^{1+\gamma}+a_2(t-a)^{2+\gamma}+...+a_n(t-a)^{n+\gamma}+..., \tag{3}$$

where a_n –the unknown coefficients that must be determined.

Substituting the power series (2), (3) into equation (1), we

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obtain:

$$\int_a^x \frac{\left(a_0(t-a)^\gamma + a_1(t-a)^{1+\gamma} + \ldots + a_n(t-a)^{n+\gamma} + \ldots\right)dt}{(x-t)^\alpha} =$$

$$= c_0 + c_1(x-a) + \ldots + c_n(x-a)^n + \ldots$$

$$\int_a^x \frac{\sum_{m=0}^{+\infty} a_m(t-a)^{m+\gamma}}{(x-t)^\alpha}dt = \left| \begin{array}{l} t = x-(x-a)z \\ dt = -(x-a)dz \\ t-a = (x-a)(1-z) \\ x-t = (x-a)z \\ t = x \to z = 0 \\ t = a \to z = 1 \end{array} \right| = \sum_{m=0}^{+\infty} a_m(x-a)^{m+\gamma-\alpha+1}\int_0^1 (1-z)^{m+\gamma} z^{-\alpha}dz = \tag{4}$$

$$= \sum_{m=0}^{+\infty} a_m(x-a)^{m+\gamma-\alpha+1} B(-\alpha+1, m+\gamma+1) = \sum_{m=0}^{+\infty} c_m\left(x-a\right)^m.$$

Here $\mathrm{B}(a,b)$ – Euler beta function, $\Gamma(a)$ – Euler gamma function.

Let $\gamma = \alpha - 1$, then, equating the terms with the same power of x in (4) yields:

$$a_m B(-\alpha+1, m+\alpha) = c_m \Rightarrow a_m = \frac{c_m}{B(-\alpha+1, m+\alpha)}.$$

If we substitute obtained coefficients in (3), then by a simple calculations we obtain:

$$\phi(x) = \sum_{k=0}^{+\infty} \frac{c_k(x-a)^{k+\alpha-1}}{B(-\alpha+1, k+\alpha)} = \sum_{k=0}^{+\infty} \frac{\Gamma(k+1)}{\Gamma(1-\alpha)\Gamma(k+\alpha)} c_k(x-a)^{k+\alpha-1} =$$

$$= \sum_{k=0}^{+\infty} \frac{k!}{\Gamma(1-\alpha)\Gamma(\alpha)(\alpha)_k} c_k(x-a)^{k+\alpha-1} = \frac{\sin\alpha\pi}{\pi}\sum_{k=0}^{+\infty} \frac{k!c_k(x-a)^{k+\alpha-1}}{(\alpha)_k} =$$

$$= \frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\sum_{k=0}^{+\infty}\frac{k!c_k(x-a)^{k+\alpha}}{(\alpha)_{k+1}} = \frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\sum_{k=0}^{+\infty} c_k(x-a)^{k+\alpha}\frac{\Gamma(\alpha)\Gamma(k+1)}{\Gamma(\alpha+k+1)} =$$

$$= \frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\left[\sum_{k=0}^{+\infty} c_k(x-a)^{k+\alpha}\int_0^1 (1-z)^m z^{\alpha-1}dz\right] = \left| \begin{array}{l} z = \dfrac{x-t}{x-a}, \quad dz = -\dfrac{dt}{x-a}, \\ 1-z = \dfrac{t-a}{x-t}, \ z = 0 \to t = x, \\ \qquad z = 1 \to t = a. \end{array} \right| =$$

$$= \frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\int_a^x \frac{\left(c_0 + c_1(t-a) + \ldots + c_n(t-a)^n + \ldots\right)dt}{(x-t)^{1-\alpha}} = \frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}$$

We are thus led to the solution:

$$\phi(x) = \frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}. \tag{5}$$

This solution is identical to the solution that is obtained in [4].

Theorem 1. If $f(t) \in C[a,b]$, then there exists a unique solution of equation Abel of first kind, which is expressed in the form (5).

The theorem is proved by direct verification that the formula (5) is a solution of equation (1).

Let's now consider generalized Abel integral equation of the second kind:

$$y(x) - \int_0^x \frac{y(t)dt}{\sqrt{x-t}} = f(x). \tag{6}$$

The solution of this equation will be sought in the form of a sum of two generalized power series:

$$y(x) = x^{\frac{1}{2}}\sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} b_n x^n \tag{7}$$

Similarly to the first case, substituting the power series (2), (7) into equation (6), performing the calculations, obtain

$$\sum_{n=0}^{+\infty} c_n x^n = x^{\frac{1}{2}}\sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} b_n x^n - \int_0^x \frac{x^{\frac{1}{2}}\sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} b_n x^n}{\sqrt{x-t}}\,dt =$$

$$= x^{\frac{1}{2}}\sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} b_n x^n - \left[\int_0^x \frac{x^{\frac{1}{2}}\sum_{n=0}^{+\infty} a_n x^n}{\sqrt{x-t}}\,dt + \int_0^x \frac{\sum_{n=0}^{+\infty} b_n x^n}{\sqrt{x-t}}\,dt\right] = \left|\begin{array}{l} t = x - xz \\ dt = -xdz \\ x - t = xz \\ t = x \to z = 0 \\ t = 0 \to z = 1 \end{array}\right| =$$

$$= x^{\frac{1}{2}}\sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} b_n x^n - \sum_{n=0}^{+\infty} a_n x^{n+1}\int_0^1 z^{-\frac{1}{2}}(1-z)^{n+\frac{1}{2}}\,dz - \sum_{n=0}^{+\infty} b_n x^{n+\frac{1}{2}}\int_0^1 z^{-\frac{1}{2}}(1-z)^n\,dz =$$

$$= x^{\frac{1}{2}}\sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} b_n x^n - \sum_{n=0}^{+\infty} a_n x^{n+1}\mathrm{B}\left(\frac{1}{2}, n+\frac{3}{2}\right) - \sum_{n=0}^{+\infty} b_n x^{n+\frac{1}{2}}\mathrm{B}\left(\frac{1}{2}, n+1\right)$$

Then by equating the terms with identical powers of x we get:

$$a_m = b_m \mathrm{B}\left(\frac{1}{2}, m+1\right),\ b_0 = c_0,$$

$$b_{m+1} = a_m \mathrm{B}\left(\frac{1}{2}, m+\frac{3}{2}\right) + c_{m+1} = b_m \mathrm{B}\left(\frac{1}{2}, m+1\right)\mathrm{B}\left(\frac{1}{2}, m+\frac{3}{2}\right) + c_{m+1},$$

$$\sum_{n=0}^{+\infty} b_n x^n = \sum_{p=0}^{+\infty}\sum_{k=0}^{p} c_k x^p \pi^{p-k}\frac{k!}{p!} = \sum_{p=0}^{+\infty}\sum_{k=0}^{p} c_k x^p \frac{\pi^{p-k}}{(p-k-1)!}\cdot\frac{\Gamma(k+1)\Gamma(p-k)}{\Gamma(p+1)} =$$

$$= \sum_{p=0}^{+\infty}\sum_{k=0}^{p} c_k x^p \frac{\pi^{p-k}}{(p-k-1)!}\cdot\mathrm{B}(k+1, p-k) = |n = p-k-1| =$$

$$= \sum_{k=0}^{+\infty} c_k x^k + \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} c_k x^{n+k+1}\frac{\pi^{n+1}}{n!}\mathrm{B}(k+1, n+1),$$

$$x^{\frac{1}{2}}\sum_{n=0}^{+\infty} a_n x^n = x^{\frac{1}{2}}\sum_{p=0}^{+\infty}\sum_{k=0}^{p} c_k x^p k!\pi^{p-k}\frac{2^{p+1}}{(2p+1)!!} =$$

$$= x^{\frac{1}{2}}\sum_{p=0}^{+\infty}\sum_{k=0}^{p} c_k x^p \pi^{p-k}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}{\Gamma\left(k+\frac{3}{2}\right)}\cdot\frac{1}{(p-k-1)!}\cdot\frac{\Gamma\left(k+\frac{3}{2}\right)\Gamma(p-k)}{\Gamma\left(p+\frac{3}{2}\right)} =$$

$$= x^{\frac{1}{2}}\sum_{p=0}^{+\infty}\sum_{k=0}^{p} c_k x^p \mathrm{B}\left(\frac{1}{2}, k+1\right)\cdot\frac{\pi^{p-k}}{(p-k-1)!}\cdot\mathrm{B}\left(k+\frac{3}{2}, p-k\right) = |n = p-k-1| =$$

$$= \sum_{k=0}^{+\infty} c_k x^{k+\frac{1}{2}}\mathrm{B}\left(\frac{1}{2}, k+1\right) + \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} c_k x^{n+k+\frac{3}{2}}\mathrm{B}\left(\frac{3}{2}, k+1\right)\frac{\pi^{n+1}}{n!}\mathrm{B}\left(k+\frac{3}{2}, n+1\right)$$

and this implies that our solution can be written as

$$y(x) = \sum_{k=0}^{+\infty} c_k x^{k+\frac{1}{2}}\mathrm{B}\left(\frac{1}{2}, k+1\right) + \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} c_k x^{n+k+\frac{3}{2}}\mathrm{B}\left(\frac{3}{2}, k+1\right)\frac{\pi^{n+1}}{n!}\mathrm{B}\left(k+\frac{3}{2}, n+1\right) +$$

$$+\sum_{k=0}^{+\infty} c_k x^k + \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} c_k x^{n+k+1} \frac{\pi^{n+1}}{n!} \mathrm{B}(k+1, n+1)$$

$$F(x) = \sum_{k=0}^{+\infty} c_k x^k + \sum_{k=0}^{+\infty} c_k x^{k+\frac{1}{2}} \mathrm{B}\left(\frac{1}{2}, k+1\right) = f(x) + \sum_{k=0}^{+\infty} x^{k+\frac{1}{2}} \int_0^1 c_k t^k (1-t)^{-\frac{1}{2}} dt =$$

$$= f(x) + \int_0^x \frac{\sum_{k=0}^{+\infty} c_k t^k}{\sqrt{x-t}} dt = f(x) + \int_0^x \frac{f(t)dt}{\sqrt{x-t}}$$

$$I = \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} c_k x^{n+k+1} \frac{\pi^{n+1}}{n!} \mathrm{B}(k+1, n+1) + \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} c_k x^{n+k+\frac{3}{2}} \mathrm{B}\left(\frac{3}{2}, k+1\right) \frac{\pi^{n+1}}{n!} \mathrm{B}\left(k+\frac{3}{2}, n+1\right) =$$

$$= \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} c_k x^{n+k+1} \frac{\pi^{n+1}}{n!} \int_0^1 t^k (1-t)^n dt + \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} c_k x^{n+k+\frac{3}{2}} \mathrm{B}\left(\frac{3}{2}, k+1\right) \frac{\pi^{n+1}}{n!} \int_0^1 t^{k+\frac{1}{2}} (1-t)^n dt =$$

$$= \int_0^1 \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} c_k x^{n+k+1} \frac{\pi^{n+1} t^k (1-t)^n}{n!} dt + \int_0^x \sum_{n=0}^{+\infty} \frac{\pi^{n+1} (x-t)^n}{n!} \sum_{k=0}^{+\infty} c_k t^{k+\frac{1}{2}} \mathrm{B}\left(\frac{3}{2}, k+1\right) dt =$$

$$= \int_0^x \sum_{n=0}^{+\infty} \frac{\pi^{n+1} (x-t)^n}{n!} \sum_{k=0}^{+\infty} c_k t^k dt + \pi \int_0^x e^{\pi(x-t)} \sum_{k=0}^{+\infty} c_k t^{k+\frac{1}{2}} \mathrm{B}\left(\frac{3}{2}, k+1\right) dt =$$

$$= \pi \int_0^x e^{\pi(x-t)} \left[f(t) + F(t) - f(t)\right] dt = \pi \int_0^x e^{\pi(x-t)} F(t) dt$$

And as a final result:

$$y(x) = f(x) + \int_0^x \frac{f(t)dt}{\sqrt{x-t}} + \pi \int_0^x e^{\pi(x-t)} F(t) dt\,. \qquad (8)$$

This solution is identical to the solution that is obtained in [4].

Theorem 2. If $f(t) \in C[a,b]$, *then there exist a unique solution of equation Abel of second kind, which is expressed in the form (7).*

As in first case, the theorem is proved by direct verification that the formula (8) is a solution of equation (6).

3. Conclusions

In this paper, solution is obtained by power series method. This may be used in more combinatorial way to obtain solution of higher degree non-linear integral equations.

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References

[1] R. Gorenflo and S. Vessella, Abel integral Equations: Analysis and Application, Springer-Verlag, Berlin-New York, 1991.

[2] Jerri, A Introduction to Integral Equations with Applications. Wiley, New York, 1999.

[3] Davis, H. T. Introduction to Nonlinear Differential and Integral Equations, 1st ed., Dover Publications, Inc., New York, 1962.

[4] A. V. Manzhirov and A. D. Polyanin, Handbook of Integral Equations: Solution Methods [in Russian], Factorial Press, Moscow, 2000.

[5] Deutsch, M, Beniaminy, I Derivative-free Inversion of AbeVs Integral Equation. Applied Physics Letters 41: pp. 27-28, 1982.

[6] Abel, NH (1826) Auflosung einer Mechanischen Aufgabe. Journal für Reine Angewandte Mathematik 1: pp. 153-157, 1826.

[7] Anderssen, RS Stable Procedures for the Inversion of AbeVs Equation. Journal of the Institute of Mathematics and its Applications 17: pp. 329-342, 1976.

[8] Minnerbo, GN, Levy, ME Inversion of AbeVs Integral Equation by Means of Orthogonal Polynomials. SIAM Journal on Numerical Analysis 6: pp. 598-616, 1969.

[9] J. D. Tamarkin, "On integrable solutions of Abel's integral equation," Annals of Mathematics, vol. 31, no. 2, pp. 219–229, 1930.

[10] A.C. Pipkin, A Course on Integral Equations, Springer–Verlag, New York, 1991.