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# Application of the GDQ Method to Vibration Analysis

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## Abstract

This paper study the vibration analysis using the differential quadrature method (DQM) which has very wide applications in the field of structural vibration of various elements such as beams, plates, cylindrical shells and tanks. One of the most advantages of the DQM is its simple forms for nonlinear formulations. In this paper, the free vibration of uniform and non-uniform beams resting on fluid layer under axial force under three sets of boundary conditions, that is, simply–simply supported (S–S), clamped–clamped supported (C–C) and clamped–simply supported (C–S) were studied using the generalized differential quadrature (GDQ). The proposed approach directly substitutes the boundary conditions into the governing equations (SBCGE). The approach of directly SBCGE is presented to overcome the drawbacks of previous approaches in treating the boundary conditions. The non-dimensional natural frequency and the normalized mode shapes of uniform and non-uniform beams were obtained. Results show good agreement with the previous analytical solutions. The effect of the varying cross section area on the vibration was studied. This work reflects the power of the DQM in solving non-uniform problems.

## Keywords

Uniform and Non-Uniform Beam, Vibration, Differential Quadrature Method

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## 1. Introduction

Often, numerical approximation methods have to be sought to solve vibration problems due to the complexity of the problems. Classical techniques such as finite element and finite difference methods are well developed and well known. These methods can provide very accurate results by using a large number of grid points; thus they are computationally expensive. In a large number of cases, only a limited number of frequencies and mode shapes or a dynamic response at only a limited number of points is needed to be found. The DQM discretizes any derivative at a point by a weighted linear sum of functional values at its neighboring points. The key to DQM is the determination of weighting coefficients. Based on the idea of integral quadrature the DQM was first introduced by Richard Bellman [1, 2]. Bellman et al. [2]

suggested two methods to determine the weighting coefficients of the first order derivative. The first method used a simple function as test functions, the second method is similar to the first one with the exception that the coordinates of grid points should be chosen as the roots of the Nth order Legendre polynomial. Unfortunately, when the order of the algebraic equation system is large, its matrix is ill-conditioned. Thus it is very difficult to obtain the weighting coefficients for a large number of grid points using this method. To overcome the drawbacks, of the above methods, Quan and Chang [12], Wen and Yu [17] use Lagrange interpolation polynomials as test function, and then obtained explicit formulations to determine the weighting coefficient for the first and second order derivatives discretization. More

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generally, Shu and Richards [14], and Shu [15] present the generalized differential quadrature (GDQ). In GDQ, the weighting coefficients of the first derivative are determined by a simple algebraic formulation, and the weighting coefficient of the higher order derivatives are determined by a recurrence relationship. The major advantage of GDQ over DQ is its ease of the computation of the weighting coefficients without any restriction on the choice of grid points. The pioneer work for the application of the DQ method to the general area of structural mechanics. To solve these equations, the boundary conditions have to be implemented appropriately. For the case where there is only one boundary condition at each boundary. However, in some cases, there is more than one boundary conditions at each boundary, which could result in difficulties in the numerical implementation of the boundary conditions. One example is the solution of the flexural vibration analysis of a thin beam or a plate. The governing equation for bending of a thin beam or a plate is a fourth order differential equation with two boundary conditions at each boundary. The details of the DQ method can be found in reference [16].

## 2. Generalized Differential Quadrature Method

In order to overcome the deficiencies which appears in classical DQM, Bellman et al. [2], a generalized differential quadrature (GDQ), which was recently proposed by Shu and Richards [14, 15] for solving partial differential equations in fluid mechanics, will be introduced and applied to solve some problems in vibration analysis. In order to find a simple algebraic expression for calculating the weighting coefficients without restricting the choice of grid meshes, Shu chose Lagrange interpolated polynomials as the set of tests functions  $g(x)$ . Shu and Richards [14, 15] gave a convenient and recurrent formula for determining the derivative weighting coefficients as follows:

$$C_{ij}^{(1)} = \frac{M_N^{(1)}(x_i)}{(x_i - x_j) \cdot M_N^{(1)}(x_j)}, \quad j \neq i \text{ and } i, j = 1, \dots, N \quad (1)$$

$$\sum_{j=1}^M C_{ij}^{(1)} = 0, \text{ for } i = 1, 2, \dots, N. \quad (2)$$

Equations (1) provide simple expressions for computing  $C_{ij}^{(1)}$  without any restriction in choice of the co-ordinates of the grid points  $x_i$ . It is obvious that  $C_{ij}^{(1)}$  can be easily calculated for  $i \neq j$ . The  $C_{ii}^{(m)}$  can be obtained from equation (2).

For the discretization of the second and higher order

derivatives, the following linear constrained relationships are applied

$$f_x^{(m)}(x_i, t) = \sum_{j=1}^N C_{ij}^{(m)} \cdot f(x_j, t). \quad (3)$$

The recurrent formula for determining the derivative weighting coefficients as follows:

$$C_{ij}^{(m)} = m \left( C_{ij}^{(1)} \cdot C_{ii}^{(m-1)} - \frac{C_{ij}^{(m-1)}}{x_i - x_j} \right), \quad j \neq i \quad i, j = 1, 2, \dots, N; \\ m = 2, 3, \dots, N-1, \quad (4)$$

$$C_{ii}^{(m)} = - \sum_{j=1, j \neq i}^N C_{ij}^{(m)} \quad \text{for } i = 1, 2, \dots, N, \quad (5)$$

where  $C_{ij}^{(m)}$  and  $C_{ij}^{(m-1)}$  are the weighting coefficients of the  $m^{\text{th}}$  and the  $(m-1)^{\text{th}}$  derivatives. The  $C_{ii}^{(m)}$  can be obtained from a relationship similar to equation (7).

Thus equations (4) and (5) together with equation (1) and equation (2) give a convenient and general form for determining the weighting coefficients for the derivatives of orders one through  $N - 1$ .

## 3. GDQ Application

The method of GDQ is used to analyze the free vibration problems of uniform and non-uniform beams resting on fluid layer under axial force in this section. In this section will calculating the natural frequencies and drawing the mode shapes of uniform and non-uniform slender beams under three sets of boundary conditions, that is, simply-simply supported (S-S), clamped-clamped supported (C-C) and clamped-simply supported (C-S). The formulations and programming are shown to be very straightforward and simple. The boundary conditions are easy to be implemented.

### Basic Equations

#### 1. Fluid Back Pressure Equations

For an incompressible, irrotational, inviscid fluid of constant density  $\gamma_f$  the pressure of the fluid  $P_f(x, y, t)$  satisfies the following equation:

$$\nabla^2 P_f + \gamma_f^2 P_f = 0, \quad (6)$$

where;

The contact conditions are:

$$\frac{\partial P_f}{\partial y} = \rho_o \frac{\partial^2 w}{\partial t^2}, \quad \text{at } y = 0 \quad (7a)$$

$$\frac{\partial P_f}{\partial y} = 0, \text{ at } y = -h. \tag{7b}$$

The initial conditions are:

$$w(x, 0, 0) = \frac{\partial w(x, 0, 0)}{\partial t}. \tag{7c}$$

The method of separation of variables is used to determine the fluid back pressure ( $P_f$ ),

$$P_f(x, y, t) = W(x)P(y)f(t). \tag{8}$$

Substituting equation (8) in equation (6) yields;

$$P(x, 0, t) = K_f \cdot W(x), \tag{9}$$

where;  $K_f(\rho_0, \gamma, \omega, h)$  is the fluid linear stiffness.

### 2. Beam Equations

The governing equation of a Bernoulli–Euler beam in bending is given by:

$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 v}{\partial x^2} \right) + \rho A \frac{\partial^2 v}{\partial t^2} + p \frac{\partial^2 v}{\partial x^2} + k_f v = 0, \quad 0 \leq x \leq L \tag{10}$$

where  $EI$  is the beam’s flexural rigidity,  $\rho A$  is the mass per unit length,  $L$  is the length of the beam and  $P_f = K_f v$  is the fluid back pressure.

Normalizing the equation (10) then the non-dimensional governing equation of a Bernoulli–Euler beam of varying cross-section resting on fluid layer under axial force may be written as:

$$S(X) \cdot \frac{d^4 W}{dX^4} + 2 \frac{dS(X)}{dX} \cdot \frac{d^3 W}{dX^3} + \frac{d^2 S(X)}{dX^2} \frac{d^2 W}{dX^2} + P \frac{d^2 W}{dX^2} + K_f W = \Omega^2 W, \tag{11}$$

where the non-dimensional coefficients is;

$$s(X) = \frac{EI}{EI_0}, \quad X = \frac{x}{L}, \quad W = \frac{V}{L_e}, \quad K_f = \frac{k_f L_e^4}{EI_0}, \quad \text{and} \\ \Omega^2 = \omega^2 \frac{\rho A L_e^4}{EI_0}.$$

Equation (11) is a 4<sup>th</sup> order ordinary differential equation with inertia ratio  $S(X) = (1 + \alpha_1 X)^{\alpha_2}$ . In the case of non-uniform beam will study two cases of inertia ratio  $S(X)$ ; the first case  $\alpha_1 = 0.5, \alpha_2 = 1.0$  and the second case  $\alpha_1 = -1.0, \alpha_2 = 1.0$ . It requires 4 boundary conditions, two at  $X = 0$ , and two at  $X = 1$ . In the present work, the following two types of boundary conditions are considered:

Simply Supported end (S)

$$W = 0 \text{ and } \frac{d^2 W(0)}{dx^2} = 0. \tag{12a}$$

Clamped Supported end (C)

$$W = 0 \text{ and } \frac{dW(0)}{dx} = 0. \tag{12b}$$

We assume that the computational domain  $0 \leq X \leq 1$  is divided by  $(N - 1)$  intervals with coordinates of grid points as  $X_1, X_2, \dots, X_N$ . With the coordinates of grid points, the GDQ weighting coefficients can be computed through equations (1), (2), (4), and (5). Then, applying the GDQ method to the equation (11) yields;

$$S(X_i) \cdot \left( \sum_{j=3}^{N-2} D_{i,j} \cdot W_j \right) + 2 \cdot S^{(1)}(X_i) \cdot \left( \sum_{j=3}^{N-2} C_{i,j} \cdot W_j \right) \\ + S^{(2)}(X_i) \cdot \left( \sum_{j=3}^{N-2} B_{i,j} \cdot W_j \right) + P \sum_{j=3}^{N-2} B_{i,j} \cdot W_j + \\ K_f \cdot W_i = \Omega^2 \cdot W_i, \quad i = 1, 2, \dots, N \tag{13}$$

where  $W_i, i = 1, 2, \dots, N$ , is the functional value at the grid  $X_i, B_{ij}, C_{ij}$  and  $D_{ij}$  is the weighting coefficient matrix of the second, third and forth order derivatives.  $S^{(2)}(X_i), S^{(1)}(X_i)$  are the second and first order derivatives of  $S(X)$  at  $X_i$ . Similarly, the derivatives in the boundary conditions can be discretized by the GDQ method. As a result, the numerical boundary conditions can be written as:

$$W_1 = 0, \tag{14a}$$

$$\sum_{k=1}^N C_{1,k}^{(n0)} \cdot W_k = 0, \tag{14b}$$

$$W_N = 0, \tag{14c}$$

$$\sum_{k=1}^N C_{N,k}^{(n1)} \cdot W_k = 0, \tag{14d}$$

where  $n0, n1$  may be taken as either 1 or 2. By choosing the value of  $n0$  and  $n1$ , Equation (20) can give the following four sets of boundary conditions,

- $n0 = 1, n1 = 1$  ——— clamped–clamped supported
- $n0 = 1, n1 = 2$  ——— clamped–simply supported
- $n0 = 2, n1 = 1$  ——— simply–clamped supported
- $n0 = 2, n1 = 2$  ——— simply–simply supported

Equations (14a) and (14c) can be easily substituted into the governing equation. This is not the case for Equations (14b) and (14d). However, one can couple these two equations together to give two solutions,  $W_2$  and  $W_{N-1}$ , as

$$W_2 = \frac{1}{AXN} \cdot \sum_{k=3}^{N-2} AXK1 \cdot W_k, \tag{15a}$$

$$W_{N-1} = \frac{1}{AXN} \cdot \sum_{k=3}^{N-2} AXKN \cdot W_k, \tag{15b}$$

where

$$AXK1 = C_{1,k}^{(n0)} \cdot C_{N,N-1}^{(n1)} - C_{1,N-1}^{(n0)} \cdot C_{N,k}^{(n1)},$$

$$AXKN = C_{1,2}^{(n0)} \cdot C_{N,k}^{(n1)} - C_{1,k}^{(n0)} \cdot C_{N,2}^{(n1)},$$

$$AXN = C_{N,2}^{(n1)} \cdot C_{1,N-1}^{(n0)} - C_{1,2}^{(n0)} \cdot C_{N,N-1}^{(n1)}.$$

According to Equations (15),  $W_2$  and  $W_{N-1}$  are expressed in terms of  $W_3, W_4, \dots, W_{N-2}$ , and can be easily substituted into the governing equation (13). It should be noted that Equation (14) provides four boundary equations. In total we have N unknowns  $W_1, W_2, \dots, W_N$ . In order to close the system, the discretized governing equation (13) has to be applied at  $(N - 4)$  mesh points. This can be done by applying Equation (13) at grid points  $X_3, X_4, \dots, X_{N-2}$ . Substituting Equations (14a), (14c) and (15) into Equation (13) gives:

$$S(X) \cdot \left( \sum_{j=3}^{N-2} D_{i,j} \cdot W_j \right) + 2 \cdot S^{(1)}(X_i) \cdot \left( \sum_{j=3}^{N-2} C_{i,j} \cdot W_j \right) + S^{(2)}(X_i) \cdot \left( \sum_{j=3}^{N-2} B_{i,j} \cdot W_j \right) + P \sum_{j=3}^{N-2} B_{i,j} \cdot W_j + K_f \cdot W_i = \Omega^2 \cdot W_i, \quad i = 3, 4, \dots, N - 2. \tag{16}$$

It is noted that Equation (16) has  $(N - 4)$  equations with

$(N - 4)$  unknowns, which can be written in matrix eigen-value form as;

$$[A]\{W\} = \Omega^2 \{W\}, \tag{17}$$

where  $\{W\} = \{W_3, W_4, \dots, W_{N-2}\}^T$ .

The Matlab program has been used to solve this eigen-value problem and get the normalized frequencies  $\Omega$  and the corresponding mode shapes.

### 4. Results and Discussion

In this section we will analyze the vibration of uniform and non-uniform beams resting on fluid layer under axial force, we will calculating the natural frequencies and corresponding mode shapes of uniform and non-uniform beams under three sets of boundary conditions, that is, simply supported–simply supported (S–S), clamped–clamped supported (C–C) and clamped–simply supported (C–S). By using the differential quadrature method, using the method of directly Substitutes the Boundary Conditions into the Governing Equations is referred to as (SBCGE).

In the present study, the coordinates of the grid points for the beam are chosen according to Chebyshev-Gauss-Lobatto by using N sampling as:

$$X(i) = \frac{1}{2} \left[ 1 - \cos \left( \frac{i-1}{N-1} \pi \right) \right], \quad i = 1, 2, 3, \dots, N$$

**Table 1a.** First three non-dimensional frequencies of uniform Simply–Simply beam

Natural Frequency	$\Omega_1$	$\Omega_2$	$\Omega_3$
Exact (Belvins [6], Qiang [13]) for uniform beam	9.8696	39.4784	88.8264
SBCGM (for uniform beam)	9.8696	39.4784	88.8249
SBCGM (for uniform beam resting on fluid, ( $K_f=1$ ))	9.9201	39.4911	88.8305
SBCGM (for uniform beam resting on fluid under axial force, ( $K_f=1, P=1$ ))	9.4095	39.9880	88.3292

**Table 1b.** First three non-dimensional frequencies of uniform Clamped–Clamped beam

Natural Frequency	$\Omega_1$	$\Omega_2$	$\Omega_3$
Exact (Belvins [6], Qiang [13]) for uniform beam	22.3733	61.6728	120.9034
SBCGM (for uniform beam)	22.3733	61.6728	120.9034
SBCGM (for uniform beam resting on fluid, ( $K_f=1$ ))	22.3956	61.6809	120.9062
SBCGM (for uniform beam resting on fluid under axial force, ( $K_f=1, P=1$ ))	22.1191	61.3064	120.4965

**Table 1c.** First three non-dimensional frequencies of uniform Clamped–Simply beam

Natural Frequency	$\Omega_1$	$\Omega_2$	$\Omega_3$
Exact (Belvins [6], Qiang [13]) for uniform beam	15.4182	49.9648	104.2477
SBCGM (for uniform beam)	15.4182	49.9648	104.2471
SBCGM (for uniform beam resting on fluid, ( $K_f=1$ ))	15.4506	49.9748	104.2519
SBCGM (for uniform beam resting on fluid under axial force, ( $K_f=1, P=1$ ))	15.0731	49.5438	103.7999

Numerical calculations have been done for both a uniform beam  $S(X) = 1$  and a non-uniform beam under three sets of boundary conditions, namely simply–simply supported (S–S), clamped–clamped supported (C–C) and clamped–simply supported (C–S). The GDQ results using the present approach for implementation boundary conditions are compared with the well known exact solution (cf. Belvins) [6] and the exact solution (Qiang) [13] for uniform beam. Table 1 lists natural frequencies for the first three modes of a uniform beam resting on fluid layer under axial force. Included in Table 1 are SBCGE results for uniform beam, the SBCGE results for uniform beam resting on fluid layer, the SBCGE results for uniform beam resting on fluid layer under axial force, the exact solution (Qiang) [13] and the exact solutions (cf. Belvins) [6] for uniform beam. The GDQ results are obtained using 15 non-uniformly spaced grid points. It can be observed from Table 1 that, the SBCGE results agree very well with the exact solutions. Also, It can be observed from Table 1 that, the natural frequencies increases when the beam resting on fluid layer while decreases when the beam resting on fluid layer under axial force.

To demonstrate better the accuracy of the solutions, the mode shapes of the first three modes of uniform beam resting on fluid layer under axial force were also obtained for three sets of boundary conditions by using 15 non-uniformly spaced grid points. The mode shapes are presented in Figures 1.

Also, Table 2, 3 lists natural frequencies of the first three modes for a non-uniform beams resting on fluid layer under axial force with different stiffness distributions  $S(X) = (1 + \alpha_1 X)^{\alpha_2}$ , the first case  $\alpha_1 = 0.5, \alpha_2 = 1.0$  and the second case  $\alpha_1 = -1.0, \alpha_2 = 1.0$ , respectively. Again, three sets of boundary conditions, (S–S, C–C and C–S), were considered for each beam. Also, the GDQ results are obtained using 15 non-uniformly spaced grid points. The varying cross section stiffness  $S(X)$  of the beams can be very easily implemented from equations (22). The computing effort is still small since one has to solve an eigen-value problem of a matrix of dimension  $11 \times 11$  only. Included in Table 2, 3 are GDQ results for non-uniform beam (Du, (1995)) if available, the SBCGE results for non-uniform beam resting on fluid layer and the SBCGE results for non-uniform beam resting on fluid layer under axial force. It can be observed from Table 2, 3 that, the natural frequencies increases when the beam resting on fluid layer while decreases when the beam resting on fluid layer under axial force.

From the tables, the convergence of this method is seen to be very good. Accurate results can be achieved by using very few grid points.

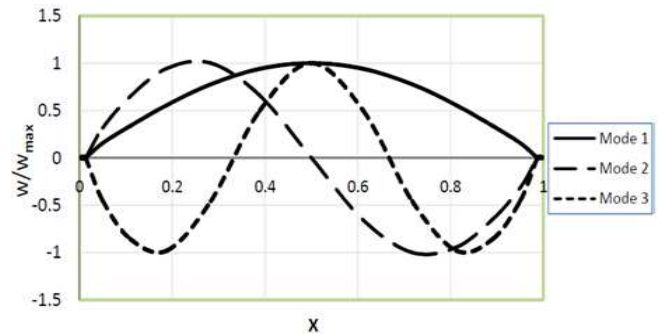


Figure (1a). First three mode shapes of uniform Simply-Simply beam ( $K_f=1.0, P=1.0$ ).

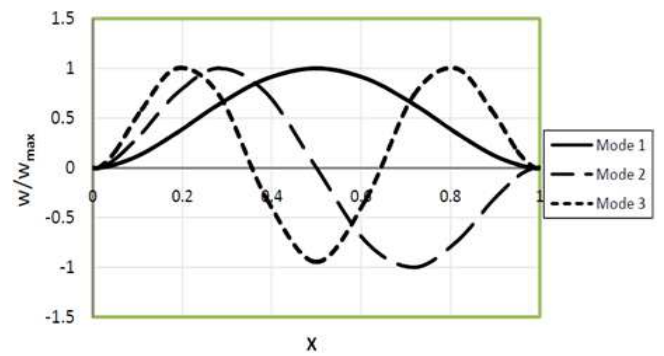


Figure (1b). First three mode shapes of uniform Clamped-Clamped beam ( $K_f=1.0, P=1.0$ ).

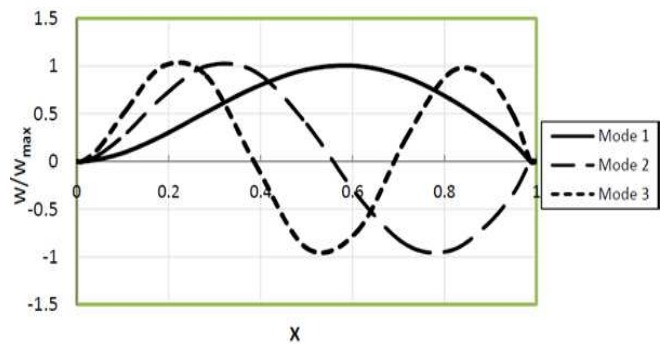


Figure (1c). First three mode shapes of uniform Clamped-Simply beam ( $K_f=1.0, P=1.0$ ).

To demonstrate better the accuracy of the solutions, the mode shapes of the first three modes of uniform beam resting on fluid layer under axial force were also obtained for three sets of boundary conditions by using 15 non-uniformly spaced grid points. The mode shapes are presented in Figures 2, 3. Figures 2 for the first case  $\alpha_1 = 0.5, \alpha_2 = 1.0$ , and Figures 3 for the second case  $\alpha_1 = -1.0, \alpha_2 = 1.0$ .

**Table 2a.** First three non-dimensional frequencies of non-uniform (S–S) beam ( $\alpha_1=0.5, \alpha_2=1$ )

Natural Frequency	$\Omega_1$	$\Omega_2$	$\Omega_3$
SBCGM (for non-uniform beam)	11.0035	43.9763	98.9180
SBCGM (for non-uniform beam resting on fluid, ( $K_f=1$ ))	11.0489	43.9877	98.9231
SBCGM (for non-uniform beam resting on fluid under axial force, ( $K_f=1, P=1$ ))	10.5923	43.5357	98.4720

**Table 2b.** First three non-dimensional frequencies of uniform (C–C) beam ( $\alpha_1=0.5, \alpha_2=1$ )

Natural Frequency	$\Omega_1$	$\Omega_2$	$\Omega_3$
SBCGM (for non-uniform beam)	24.8883	68.6339	134.5731
SBCGM (for non-uniform beam resting on fluid, ( $K_f=1$ ))	24.9084	68.6412	134.5768
SBCGM (for non-uniform beam resting on fluid under axial force, ( $K_f=1, P=1$ ))	24.6599	68.3046	98.4720

**Table 2c.** First three non-dimensional frequencies of uniform (C–S) beam ( $\alpha_1=0.5, \alpha_2=1$ )

Natural Frequency	$\Omega_1$	$\Omega_2$	$\Omega_3$
SBCGM (for non-uniform beam, Du [10])	16.9280	55.4090	115.4800
SBCGM (for non-uniform beam)	16.9279	55.3994	115.8263
SBCGM (for non-uniform beam resting on fluid, ( $K_f=1$ ))	16.9574	55.4085	115.8306
SBCGM (for non-uniform beam resting on fluid under axial force, ( $K_f=1, P=1$ ))	16.6218	55.0250	115.4272

**Table 3a.** First three non-dimensional frequencies of non-uniform (S–S) beam ( $\alpha_1=-1, \alpha_2=1$ )

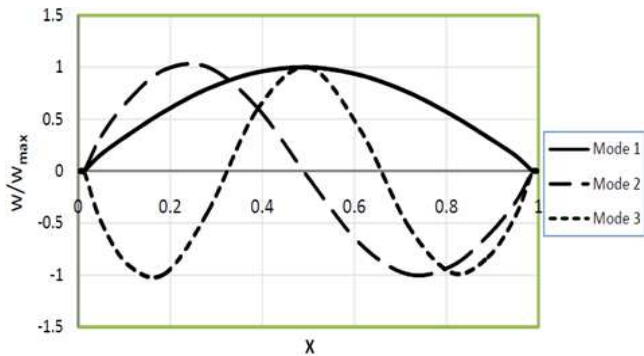
Natural Frequency	$\Omega_1$	$\Omega_2$	$\Omega_3$
SBCGM (for non-uniform beam)	6.3249	24.6150	53.5371
SBCGM (for non-uniform beam resting on fluid, ( $K_f=1$ ))	6.4035	24.6353	53.5465
SBCGM (for non-uniform beam resting on fluid under axial force, ( $K_f=1, P=1$ ))	5.5306	23.7378	52.5812

**Table 3b.** First three non-dimensional frequencies of non-uniform (C–C) beam ( $\alpha_1=-1, \alpha_2=1$ )

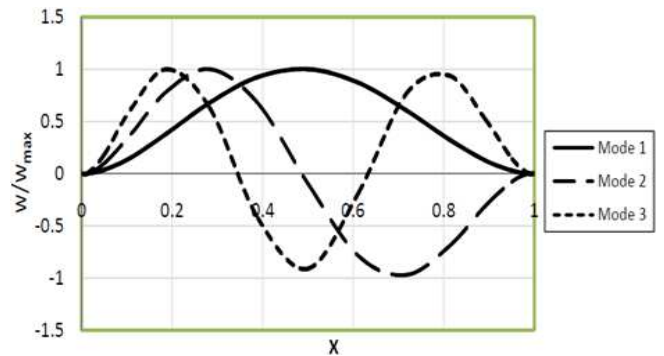
Natural Frequency	$\Omega_1$	$\Omega_2$	$\Omega_3$
SBCGM (for non-uniform beam)	11.9742	34.3839	68.3635
SBCGM (for non-uniform beam resting on fluid, ( $K_f=1$ ))	12.0159	34.3984	68.3708
SBCGM (for non-uniform beam resting on fluid under axial force, ( $K_f=1, P=1$ ))	11.4527	33.6574	67.5535

**Table 3c.** First three non-dimensional frequencies of non-uniform (C–S) beam ( $\alpha_1=-1, \alpha_2=1$ )

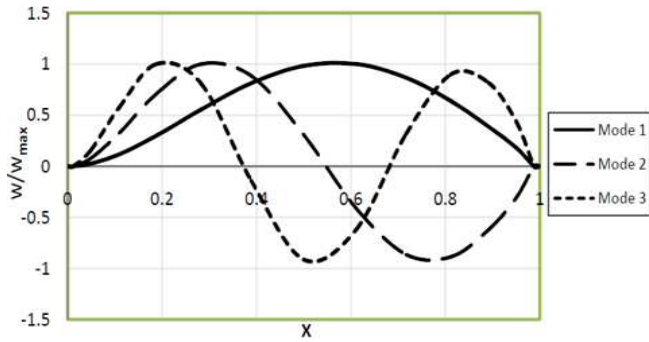
Natural Frequency	$\Omega_1$	$\Omega_2$	$\Omega_3$
SBCGM (for non-uniform beam, Du [10])	10.7390	31.6930	65.3110
SBCGM (for non-uniform beam)	10.3991	31.1571	63.4720
SBCGM (for non-uniform beam resting on fluid, ( $K_f=1$ ))	10.4471	31.1731	63.4799
SBCGM (for non-uniform beam resting on fluid under axial force, ( $K_f=1, P=1$ ))	9.7553	30.3305	62.5810



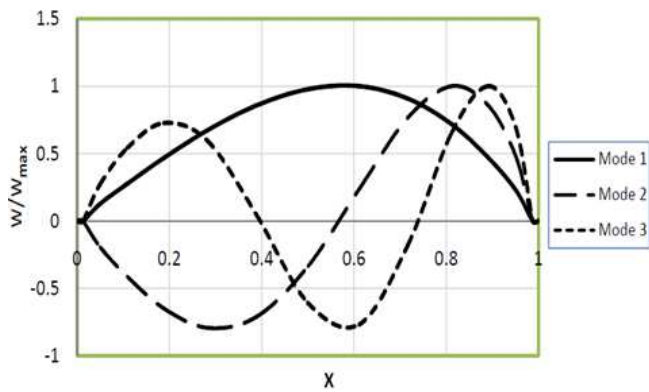
**Figure (2a).** First three mode shapes of non-uniform Simply-Simply beam ( $K_f=1.0, P=1.0, \alpha_1=0.5, \alpha_2=1.0$ ).



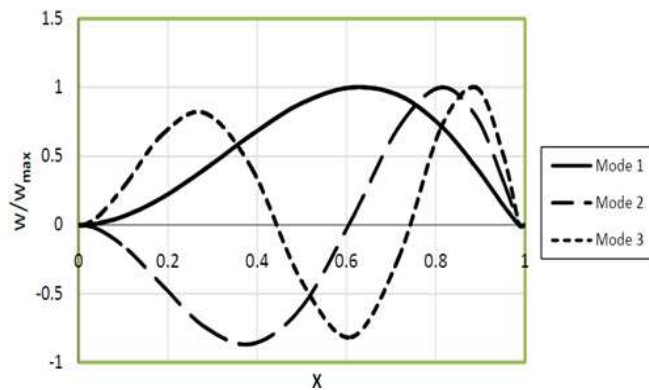
**Figure (2b).** First three mode shapes of non-uniform Clamped-Clamped beam ( $K_f=1.0, P=1.0, \alpha_1=0.5, \alpha_2=1.0$ ).



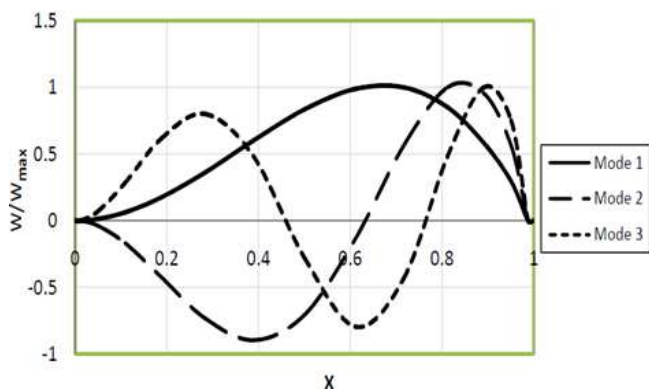
**Figure (2c).** First three mode shapes of non-uniform Clamped-Simply beam ( $K_f=1.0$ ,  $P=1.0$ ,  $\alpha_1=0.5$ ,  $\alpha_2=1.0$ ).



**Figure (3a).** First three mode shapes of non-uniform Simply-Simply beam ( $K_f=1.0$ ,  $P=1.0$ ,  $\alpha_1=-1.0$ ,  $\alpha_2=1.0$ ).



**Figure (3b).** First three mode shapes of non-uniform Clamped-Clamped beam ( $K_f=1.0$ ,  $P=1.0$ ,  $\alpha_1=-1.0$ ,  $\alpha_2=1.0$ ).



**Figure (3c).** First three mode shapes of non-uniform Clamped-Simply beam ( $K_f=1.0$ ,  $P=1.0$ ,  $\alpha_1=-1.0$ ,  $\alpha_2=1.0$ ).

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