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Fixed Point Theorems in Fuzzy Normed Spaces for Weak Contractive Mappings

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Abstract

We define fixed point for the class of weak contractive type mappings in fuzzy normed spaces. We obtain several convergence theorems for fixed points by means of Picard iteration on fuzzy normed spaces. These extend the corresponding results in literature by providing error estimates, rate of convergence for the used iterative method as well as results concerning the data dependence of the fixed points on fuzzy norm spaces.

Keywords

Fuzzy Normed Space, Fuzzy Fixed Point, Weak Contraction, Convergence Theorem

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1. Introduction

Nowadays, fixed point play an important role in different areas of mathematics, and its applications, particularly in mathematics, physics, differential equation. For details, one can refer to, Amann, [1] Franklin, [2] Mohsenalhosseini et al. [3]. Since fuzzy mathematics and fuzzy physics along with the classical ones are constantly developing, the fuzzy type of the fixed point can also play an important role in the new fuzzy area. Some mathematicians have defined fuzzy norms on a vector space from various points of view [4], [5].

Chitra and Mordeson [6] introduce a definition of norm fuzzy and thereafter the concept of fuzzy norm space has been introduced and generalized in different ways by Bag and Samanta in [7], [8], [9].

In this paper, starting from the article of Berinde [10], we study some well-known contractive type mappings on fuzzy normed spaces, and we give some fuzzy approximate fixed points of such mappings.

2. Some Preliminary Results

Throughout this article, the symbols $\wedge$ and $\vee$ mean the min and the max, respectively. We now start our work with the following:

Definition 2.1. [7] Let U be a linear space on R. A function $N: U\times R \to [0,1]$ is called fuzzy norm if and only if for every $x,u \in U$ *and for every* $c\in R$ the following properties are satisfied:

(F1) : $N(x,t)=0$ for every $t\in R^{-}\bigcup\{0\}$,

(F2) : $N(x,t)=1$ if and only if $x=0$ for every $t\in R^{+}$,

(F3) : $N(cx,t)=N(x,\frac{t}{|c|})$ for every $c=0$ and $t\in R^{+}$,

(F4) : $N(x+t,s+t)\geq \min\{N(x,t),N(u,t)\}$ for every $s,t\in R^{+}$,

(F5) : the function $N(x,.)$ is nondecreasing on R

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and $\lim_{t\to\infty} N(x,t)=1$.

(F6) : $\forall \ t\in R^{+}\ N(x,t)>0 \Rightarrow x=0$.

(F7) : The function $N(x,.)$ is continuous for every $x\neq 0$, and on subset $\{t: 0<N(x,t)<1\}$ is strictly increasing.

Let $N(U,N)$ be a fuzzy norm space. For all $\alpha\in(0,1)$, we define α -norm on U as follows:

$$\|x\|_{\alpha} = \wedge\{t>0: N(x,t)\geq\alpha\} \tag{1}$$

for every $x\in U$.

Definition 2.2. [11] Let (U,N) be a fuzzy normed space, $T:U\to U$, $\varepsilon>0$ and $u_0\in U$. The $u_0\in U$ is a F^z - approximate fixed point (fuzzy approximate fixed point) of T if for some $\alpha\in(0,1)$

$$\wedge\{t>0: N(u_0-Tu_0,t)\geq\alpha\}\leq\varepsilon.$$

Definition 2.3. [11] A mapping $T:U\to U$, is a F-Kannan operator if there exists $a\in(0,\frac{1}{2})$ such that

$$\begin{aligned}&\wedge\{t>0: N(Tu-Tv,t)\geq\alpha\}\leq\\&\wedge a[\{t>0: N(u-Tu,t)\geq\alpha\}+\\&\wedge\{t>0: N(v-Tv,t)\geq\alpha\}],\end{aligned}$$

for all $u,v\in U$.

3. Fuzzy Fixed Point

Definition 3.1. A mapping $T:U\to U$, is called F^z - weak contraction or (δ,L) -contraction if there exist a constant $\delta\in(0,1)$ and some $L\geq 0$ such that

$$\begin{aligned}&\wedge\{t>0: N(Tu-Tv,t)\geq\alpha\}\leq\\&\delta\wedge\{t>0: N(u-v,t)\geq\alpha\}+\\&L\wedge\{t>0: N(u-Tu,t)\geq\alpha\},\end{aligned} \tag{2}$$

for all $u,v\in U$.

Remark 3.2. Due to the symmetry of the distance, the weak contraction condition (2) implicitly includes the following dual one

$$\begin{aligned}&\wedge\{t>0: N(Tu-Tv,t)\geq\alpha\}\leq\\&\delta\wedge\{t>0: N(u-v,t)\geq\alpha\}+\\&L\wedge\{t>0: N(u-Tv,t)\geq\alpha\},\end{aligned} \tag{3}$$

for all $u,v\in U$.

Remark 3.3. In the rest of the paper we will denote the set of all F^z -fixed points of T, by

$$F^z(T)=\wedge\{u\in U: N(u-Tu,t)\geq\alpha\}=0, \tag{4}$$

for some $\alpha\in(0,1)$.

Proposition 3.4. Let (U,N) be a fuzzy normed space and $T:U\to U$, is an F^z -weak contraction. Then

1) $F^z(T)\neq\varphi$;

2) The Picard iteration $\{u_n\}_{n=0}^{\infty}$ given by

$$u_{n+1}=Tu_n \qquad n=0,1,2,...$$

converges to u^* for any $u_0\in U$.

3) The following estimates

$$\text{A)}\quad\begin{aligned}&\wedge\{t>0: N(u_n-u^*,t)\geq\alpha\}\leq\\&\frac{\delta^n}{1-\delta}\wedge\{t>0: N(u_0-u_1,t)\geq\alpha\}\end{aligned} \tag{5}$$

for $n=0,1,2,...$

$$\text{B)}\quad\begin{aligned}&\wedge\{t>0: N(u_n-u^*,t)\geq\alpha\}\leq\\&\frac{\delta}{1-\delta}\wedge\{t>0: N(u_{n-1}-u_n,t)\geq\alpha\}\end{aligned} \tag{6}$$

for $n=1,2,...$

hold, where $\delta\in(0,1)$ is constant.

Proof: Let $\{u_n\}_{n=0}^{\infty}$ be the Picard iteration, starting from $u_0\in U$. arbitrary. Then by (2) we have

$$\begin{aligned}&\wedge\{t>0: N(u_n-u_{n+1},t)\geq\alpha\}=\\&\{t>0: N(Tu_{n-1}-Tu_n,t)\geq\alpha\}\leq\\&\delta\wedge\{t>0: N(u_{n-1}-u_n,t)\geq\alpha\}+\\&L\wedge\{t>0: N(u_n-Tu_{n-1},t)\geq\alpha\}.\end{aligned}$$

By the Picard iteration of $\{u_n\}_{n=0}^{\infty}$, we have

$$\begin{aligned}&\wedge\{t>0: N(u_n-u_{n+1},t)\geq\alpha\}=\\&\wedge\{t>0: N(Tu_{n-1}-Tu_n,t)\geq\alpha\}\leq\\&\delta\wedge\{t>0: N(u_{n-1}-u_n,t)\geq\alpha\}.\end{aligned}$$

By induction

$$\wedge\{t>0:N(u_{n+k}-u_{n+k+1},t)\geq\alpha\}\leq$$
$$\delta^k\wedge\{t>0:N(u_{n-1}-u_n,t)\geq\alpha\}.,k\in N^*$$

and hence

$$\wedge\{t>0:N(u_{n+p}-u_n,t)\geq\alpha\}\leq$$
$$\wedge(\delta+\delta^2+...+\delta^p)\times\wedge\{t>0:N(u_n-Tu_{n-1},t)\geq\alpha\}$$

which yields

$$\wedge\{t>0:N(u_{n+p}-u_n,t)\geq\alpha\}\leq$$
$$\frac{\delta(1-\delta^p)}{1-\delta}\wedge\{t>0:N(u_n-u_{n-1},t)\geq\alpha\} \tag{7}$$

all $n,p\in N^*$. Since,

$$\wedge\{t>0:N(u_n-u_{n-1},t)\geq\alpha\}\leq$$
$$\delta^{n-1}\wedge\{t>0:N(u_0-u_1,t)\geq\alpha\},\qquad n\geq 1$$

from (7) we obtain

$$\wedge\{t>0:N(u_{n+p}-u_n,t)\geq\alpha\}\leq$$
$$\frac{\delta^n(1-\delta^p)}{1-\delta}\wedge\{t>0:N(u_0-u_1,t)\geq\alpha\} \tag{8}$$

Now by letting $p\to\infty$ in (8) and (7) we obtain the estimates (5) and (6), respectively.

Proposition 3.5. Let (U,N) be a fuzzy normed space and $T:U\to U$, is an F^z-weak contraction for which there exist $\theta\in(0,1)$ and some $L_1\geq 0$ such that

$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq$$
$$\delta\wedge\{t>0:N(u-v,t)\geq\alpha\}+$$
$$L_1\wedge\{t>0:N(u-Tv,t)\geq\alpha\} \tag{9}$$

for all $u,v\in U$. Then

1) T has a unique F^z-fixed point, i.e. $F^z(T)=\{u^*\}$;

2) The Picard iteration $\{u_n\}_{n=0}^{\infty}$ given by

$$u_{n+1}=Tu_n\qquad n=0,1,2,...$$

converges to u^* for any $u_0\in U$.

3) The a priori and a posteriori error estimates

A)
$$\wedge\{t>0:N(u_n-u^*,t)\geq\alpha\}\leq$$
$$\frac{\theta^n}{1-\theta}\wedge\{t>0:N(u_0-u_1,t)\geq\alpha\} \tag{10}$$

for $n=0,1,2,...$

B)
$$\wedge\{t>0:N(u_n-u^*,t)\geq\alpha\}\leq$$
$$\frac{\theta}{1-\theta}\wedge\{t>0:N(u_{n-1}-u_n,t)\geq\alpha\} \tag{11}$$

for $n=0,1,2,...$

hold.

4) The rate of convergence of the Picard iteration is given by

$$\wedge\{t>0:N(u_n-u^*,t)\geq\alpha\}\leq$$
$$\theta\wedge\{t>0:N(u_{n-1}-u^*,t)\geq\alpha\} \tag{12}$$

Proof: Let $\{u_n\}_{n=0}^{\infty}$ be the Picard iteration, starting from $u_0\in U$. arbitrary. Then by (2) we have

$$\wedge\{t>0:N(u_n-u_{n+1},t)\geq\alpha\}=$$
$$\{t>0:N(Tu_{n-1}-Tu_n,t)\geq\alpha\}\leq$$
$$\delta\wedge\{t>0:N(u_{n-1}-u_n,t)\geq\alpha\}+$$
$$L_1\wedge\{t>0:N(u_n-Tu_{n-1},t)\geq\alpha\}.$$

By the Picard iteration of $\{u_n\}_{n=0}^{\infty}$, we have

$$\wedge\{t>0:N(u_n-u_{n+1},t)\geq\alpha\}=$$
$$\wedge\{t>0:N(Tu_{n-1}-Tu_n,t)\geq\alpha\}\leq$$
$$\theta\wedge\{t>0:N(u_{n-1}-u_n,t)\geq\alpha\}.$$

By induction

$$\wedge\{t>0:N(u_{n+k}-u_{n+k+1},t)\geq\alpha\}\leq$$
$$\theta^k\wedge\{t>0:N(u_{n-1}-u_n,t)\geq\alpha\}.,k\in N^*$$

and hence

$$\wedge\{t>0:N(u_{n+p}-u_n,t)\geq\alpha\}\leq$$
$$\wedge(\theta+\theta^2+...+\theta^p)\times\wedge\{t>0:N(u_n-Tu_{n-1},t)\geq\alpha\}$$

which yields

$$\wedge\{t>0:N(u_{n+p}-u_n,t)\geq\alpha\}\leq$$
$$\frac{\theta(1-\theta^p)}{1-\theta}\wedge\{t>0:N(u_n-u_{n-1},t)\geq\alpha\} \tag{13}$$

for all $n,p\in N^*$. Since,

$$\wedge\{t>0:N(u_n-u_{n-1},t)\geq\alpha\}\leq$$
$$\theta^{n-1}\{t>0:N(u_0-u_1,t)\geq\alpha\},\qquad n\geq 1$$

from (13) we obtain

$$\wedge\{t>0:N(u_{n+p}-u_n,t)\geq\alpha\}\leq \frac{\theta^n(1-\theta^p)}{1-\theta}\wedge\{t>0:N(u_0-u_1,t)\geq\alpha\} \tag{14}$$

Now by letting $p\to\infty$ in (14) and (13) we obtain the estimates (10) and (11), respectively.

Again, by (2) we have

$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq \theta\wedge\{t>0:N(u-v,t)\geq\alpha\}+ L_1\wedge\{t>0:N(u-Tv,t)\geq\alpha\} \tag{15}$$

where $\theta=\frac{a}{1-a}, L_1=\frac{2a}{1-a}$.

Take $u:=u^*$ $v:=u_{n-1}$ in (15) to obtain

$$\wedge\{t>0:N(u_n-u^*,t)\geq\alpha\}\leq \frac{a}{1-a}\wedge\{t>0:N(u_{n-1}-u^*,t)\geq\alpha\}$$

that is, the estimate (12).

Remark 3.6. Note that, by the symmetry of the distance, (9) is satisfied for all $u,v\in U$ if and only if

$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq \theta\wedge\{t>0:N(u-v,t)\geq\alpha\}+ L_1\wedge\{t>0:N(u-Tv,t)\geq\alpha\}, \tag{16}$$

also holds, for all $u,v\in U$.

Corollary 3.7. Let (U,N) be a fuzzy normed space and $T:U\to U$, is an F-Kannan operator Then

1) T has a unique F^z -fixed point, i.e. $F^z(T)=\{u^*\}$;

2) The Picard iteration $\{u_n\}_{n=0}^{\infty}$ given by

$$u_{n+1}=Tu_n \qquad n=0,1,2,...$$

converges to u^* for any $u_0\in U$.

3) The a priori and a posteriori error estimates

$$\wedge\{t>0:N(u_n-u^*,t)\geq\alpha\}\leq \frac{\delta^n}{1-\delta}\wedge\{t>0:N(u_0-u_1,t)\geq\alpha\}$$

for $n=0,1,2,...$

$$\wedge\{t>0:N(u_n-u^*,t)\geq\alpha\}\leq \frac{\delta}{1-\delta}\wedge\{t>0:N(u_{n-1}-u_n,t)\geq\alpha\}$$

for $n=0,1,2,...$

hold.

4) The rate of convergence of the Picard iteration is given by

$$\wedge\{t>0:N(u_n-u^*,t)\geq\alpha\}\leq \delta\wedge\{t>0:N(u_{n-1}-u^*,t)\geq\alpha\}.$$

Proof: By Definition 2.3. and triangle rule, we get

$$\begin{aligned}&\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq\\ &a[\wedge\{t>0:N(u-Tu,t)\geq\alpha\}+\\ &\wedge\{t>0:N(v-Tv,t)\geq\alpha\}]\leq\\ &\{\ [\wedge\{t>0:N(u-v,t)\geq\alpha\}+\\ &\wedge\{t>0:N(v-Tu,t)\geq\alpha\}]+\\ &[\wedge\{t>0:N(v-Tu,t)\geq\alpha\}+\\ &\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}]\ \}\end{aligned}$$

Therefore

$$(1-a)\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq a\wedge\{t>0:N(u-v,t)\geq\alpha\}+ 2a\wedge\{t>0:N(v-Tu,t)\geq\alpha\}.$$

Then

$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq \frac{a}{1-a}\wedge\{t>0:N(u-v,t)\geq\alpha\}+ \frac{2a}{1-a}\wedge\{t>0:N(v-Tu,t)\geq\alpha\},$$

for all $u,v\in U$, i.e., in view of $a\in(0,\frac{1}{2})$,

Definition 3.1. holds with $\delta=\frac{a}{1-a}, L=\frac{2a}{1-a}$.

Since F-Kannan operator is symmetric with respect to u and v, (3) also holds. In a similar way, by the same Definition 2.3. and triangle rule, we get

$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq$$
$$a[\wedge\{t>0:N(u-Tu,t)\geq\alpha\}+$$
$$\wedge\{t>0:N(v-Tv,t)\geq\alpha\}]\leq$$
$$\{\ \wedge\{t>0:N(u-Tu,t)\geq\alpha\}+$$
$$[\wedge\{t>0:N(v-u,t)\geq\alpha\}+$$
$$\wedge\{t>0:N(u-Tu,t)\geq\alpha\}+$$
$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}]\ \}$$

Therefore

$$(1-a)\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq$$
$$a\wedge\{t>0:N(u-v,t)\geq\alpha\}+$$
$$2a\wedge\{t>0:N(u-Tu,t)\geq\alpha\}.$$

Then

$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq$$
$$\frac{a}{1-a}\wedge\{t>0:N(u-v,t)\geq\alpha\}+$$
$$\frac{2a}{1-a}\wedge\{t>0:N(u-Tu,t)\geq\alpha\},$$

for all $u,v\in U$, which shows that (9) and (16) hold with $\theta=\frac{a}{1-a}, L_1=\frac{2a}{1-a}$.

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