What is Mathematics - an Overview

Liaqat Ali Khan[*]

Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

Abstract

Mathematics is based on deductive reasoning though man's first experience with mathematics was of an inductive nature. This means that the foundation of mathematics is the study of some logical and philosophical notions. We elaborate in simple terms that the deductive system involves four things: (1) A set of primitive undefined terms; (2) Definitions evolved from the undefined terms; (3) Axioms or postulates; (4) Theorems and their proofs. We also include some historical remarks on the nature of mathematics.

1. Introduction

Mathematics is not only concerned with everyday problems, but also with using imagination, intuition and reasoning to find new ideas and to solve puzzling problems. One method used by mathematicians in discovering new ideas is to perform experiments. This is called the "experimental method" or "inductive reasoning". When a scientist takes a large number of careful observations and from them infers some probable results or when he repeats an experiment many times and from these data arrives at some probable conclusion, he is using inductive reasoning. That is to say, from a large number of specific cases he obtains a single general inference.

The other method is based on reasoning rather than on experiments or observations. This is called "deductive reasoning". When a mathematician begins with a set of acceptable conditions, called the hypothesis and by a series of logical implications reaches a valid conclusion, he employs deductive reasoning. The major difference in the two methods is implied in the two words: "probable" with respect to inductive reasoning and `valid' relative to deductive reasoning. For example, if we perform an experiment successfully say a thousand times, then another twenty successful trails would lend credence to the result, but we have no assurance whatever that the experiment will not fall on the very next trail. On the other hand in a deductive system, once we accept the hypothesis, the validity of our conclusion is inevitable provided each implication in the reasoning process is a logical consequence of what which proceeds it. Here "consistency" of a logical system means that no theorem of the system contradicts another and "validity" means that the system's rules of proof never allow a false inference from true premises.

2. Deductive Reasoning System

As mentioned above, mathematics is based on deductive reasoning though man's first experience with mathematics was of an inductive nature. The ancient Egyptians and Babylonians developed many mathematical ideas through observation and experimentation and made use of this mathematics in their daily life. Then the Greeks became

* Corresponding author
E-mail address: akliaqat@gmail.com

interested in philosophy and logic and placed a great emphasis on reasoning. For example, in Geometry, the axiomatic development was first developed by them from 500 to 300 BC, and was described in detail by Euclid around 300 BC. They accepted a few most basic mathematical assumptions and used them to prove deductively most of the geometric facts we know today. Our high school geometry is an excellent example of a deductive system. Recall that in the study of geometry, we began with a set of undefined terms, such as point, line etc. We then made some definitions, for example, those of angle, parallel lines, perpendicular lines, triangle etc. Next, we listed a number of statements concerning these undefined and defined terms which we accepted to be true without proof; these assumptions we called, axioms or postulates. Finally, we were able to prove a considerable number of propositions or theorems by deductive reasoning.

In summary, we observe that the study of foundations of mathematics involves an abstract deductive system consisting of:

1. A set of primitive undefined terms;

2. Definitions evolved from the undefined terms;

3. Axioms or postulates;

4. Theorems and their proofs.

We now discuss each of them as follow.

UNDEFINED TERMS: To build a mathematical system based on logic, the mathematician begins by using some words to express their ideas, such as `number' or a `point'. These words are undefined and are sometimes called `primitive terms'. These words usually have some meaning because of experience we have had with them. It may seem strange that in mathematics, a field with which precision and accuracy are commonly associated, we do not (and cannot) `start from scratch' but find it necessary to begin with a set of undefined terms. Why do we not start with precise definitions? An attempt to define any of the fundamental undefined terms, such as point, set, number or element demonstrate that we are soon led to what is referred to as 'circular reasoning'. For example, let us try to define `point'. What is a `point'? A point is a position at which something exists. But what is meant by position? The location of an object, naturally. But what does location of an object imply: a point. So we are back where we started. In a like manner, an attempt to define any of the other undefined terms of mathematics would also result in circular reasoning. Hence, it should now be clear that the use of primitive terms is indispensable because they serve as the foundation upon which the system rests.

For obvious reasons, the primitive terms, a mathematician chooses must be simple in form and as small in number as possible. They usually appeal to the intuition, more or less, but it is important to distinguish the intuitive ideas behind them and the part they play in the theory. It is not completely true to say that a primitive term has no formal meaning. It may have content because of the logical position we put in it.

DEFINITIONS: A definition, Bertrand Russell says, is a declaration that a certain newly introduced term or combination of terms is to mean the same as a certain other combination of terms, of which the meaning is already known. It assigns a meaning to a term by means of primitive terms and terms already defined.

It is to be observed that although we employ definitions, yet "definitions" does not appear among our primitive ideas because, strictly speaking, the definitions are no part of our subject. Practically, of course, if we introduce no definitions, our formulae would very soon become so lengthy, as to be unimaginable.

In spite of the fact that definitions are theoretically superfluous, it is nevertheless true, that they often convey more important information than is contained in the proposition in which they are used. Definitions clarify and simplify expressions. We need to define our terms so that we can use short names for complex ideas. Also definitions contain an analysis of a common idea and can, therefore, classify, that we wish to single out quadrilaterals with opposite parallel sides. We may do this by means of a definition: "a parallelogram is a quadrilateral whose opposite sides are parallel". If we assume in this definition that `quadrilateral', `opposite sides' and `parallel' have been defined previously, then what we have done is to define the class of parallelograms.

AXIOMS AND POSTULATES: At the start of every mathematical theory (such as Real Numbers System, Group Theory, Topology, Quantum Mechanics), some kinds of foundations are needed. For this purpose, a set of independent fundamental statements is asserted. These assertions are called axioms and postulates. Both the axioms and the postulates have their roots in antiquity. To quote Aristotle, "Every demonstrative science must start from indemonstrable principles. Otherwise, the steps of demonstration would be endless". Both the axioms and the postulates presumably are principles, so clearly true that we accept them without a corresponding proof. In Euclid's time (300 BC), axioms referred specifically to an assumption in geometry. Today the distinction is disregarded and both terms are used interchangeably.

The axioms of a mathematical theory are usually stipulated at the beginning of the theory, immediately after announcing the primitive terms. These terms are the bricks with which we

build up these axioms. The axioms may contain such statements as: "Things equal to the same thing are equal to one another". "Every line is a set of points". They are necessary if we are to avoid an infinite regression which would certainly result if we only accepted what we could prove. Once the axioms have been chosen, we become more severe about the subsequent propositions.

THEOREMS AND THEIR PROOFS: A `theorem' is a statement whose truth is established by formal proof. The bulk of any branch of mathematics consists of the collection of theorems that pertain to that particular area. Much of the beauty of mathematics lies in the sequential development of the subject through the proofs of its theorems. A `proof' is a chain of reasoning that succeeded in establishing a conclusion by showing that it follows logically from premises that already are known to be true. In proving a theorem, we may use our undefined and defined terms, and our axioms and of course any theorem we prove, the more knowledge we have at our disposal to prove additional theorems. In any mathematical theory, to prove the first theorem, A (say), the only arguments that can be used are the axioms. And to prove a second theorem, B (say), we may use the axioms and Theorem A and similarly for the subsequent theorems. Hence we state the principle: "a proof demonstrates the validity of a proposition using an argument based entirely on the axioms and the previously established theorems".

Kinds of Proofs: There are two kinds of proof: direct proof and indirect proof. Most theorems have the form "a statement p implies another statement q". To demonstrate such a statement we proceed with an assumption usually called the hypothesis in the following ways:

Assert p (i.e. suppose p is given). From this we construct a demonstration that ends with the statement q.

This program makes up what we called a "direct proof".

The `indirect proof', also called "proof by contradiction" (reductio ad absurdum, in Latin), depends essentially on the notion of negation. This idea can be stated in the following form:

"To prove the Theorem A indirectly, we affirm its negation. From this we construct an argument that concludes with the negation of a result already known to be true. This is a legitimate proof of Theorem A".

Truth of Assertions: We have spoken of the truth of certain assertions. What does the word `truth' mean in this context? A proposition is true if it can be proved by means of the axioms and theorems proved previously. Notice that a theorem may be true in one theory but false in another; it all depends on the initial axioms. In "plane geometry", the

statement that the sum of the angles of a triangle is two right angles is true, but it is no longer true in "Riemannian geometry". In "classical mechanics" mass is indestructible, but in "quantum theory", a mass can be destroyed and replaced by energy.

3. Some Remarks and Quotes

A. The above described Deductive System is also called *Formalism*. In fact, there are two dominant schools of thought about the nature of Mathematics: one is the Platonist or Realist (deriving from Plato) and the other is Formalist. The Platonists believe that mathematical objects exist independent of us and inhabit a world of their own. They are not invented by us but rather discovered.

Formalists on the other hand believe that there are no such things as mathematical objects. Mathematics consists of definitions, axioms and theorems invented by mathematicians and have no meaning in themselves except that which we ascribe to them. This school of thought was introduced by David Hilbert in 1921.

During the 1920's shock waves had run through the science of physics, because of Heisenberg's Uncertainty Principle (introduced first by the German physicist Werner Heisenberg in 1927). This principle states that you can never simultaneously know the exact position and the exact speed of an object. In 1931, a 25 year old Austrian mathematician Kurt Gödel shocked the worlds of mathematics and philosophy by showing that there are mathematical truths which simply cannot be proved.

Kurt Gödel (1906-1978) was regarded as a brilliant mathematician and perhaps the greatest logician since Aristotle. His famous "incompleteness theorem" was a fundamental result about axiomatic systems, showing that in any axiomatic mathematical system, there are propositions that cannot be proved or disproved within the axioms of the system. In particular the consistency of the axioms cannot be proved. This ended a hundred years of attempts to establish axioms which would put the whole of mathematics on an axiomatic basis. These included some major attempts by several logicians and mathematicians of that time (such as Germans' Richard Dedekind (1831-1916), Georg Cantor (1845-1918), Friedrich Frege (1848-1925), David Hilbert (1862-1943), Ernst Zermelo (1871-1953), Italian's Giuseppe Peano (1858-1932),), Dutch L.E.J. Brouwer (1881-1966), British Bertrand Russell (1872-1970)).

B. Gödel's results did not destroy the fundamental idea of formalism, but it did demonstrate that any system would have to be more comprehensive than that envisaged by

Hilbert and others. In fact, these results were a landmark in 20th-century mathematics, showing that mathematics is not a finished object, as had been believed. It also implies that a computer can never be programmed to answer all mathematical questions. Among physicists, Gödel is known as the man who proved that time travel to the past was possible under Einstein's equations.

C. Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true – (Bertrand Russell, Mysticism and Logic (1917) Ch. 4).

D. "Obvious" is the most dangerous word in mathematics (E.T. Bell, 1883-1960).

E. To arrive at the simplest truth, as Newton knew and practiced, requires years of contemplation. Not activity. Not reasoning. Not calculating. Not busy behaviour of any kind. Not reading. Not talking. Not making an effort. Not thinking. Simply bearing in mind what it is one needs to know. (George Spencer Brown, 1923-)

F. Pure mathematics is on the whole distinctly more useful than applied. For what is useful above all is technique, and mathematical technique is taught mainly through pure mathematics. (G.H. Hardy, 1877-1947)

G. As one ancient stated, teaching is not a matter of pouring knowledge from one mind into another as one pours water from one glass into another. It is more like one candle igniting another. Each candle burns with its own fuel.

H. For more than two thousand years some familiarity with mathematics has been regarded as an indispensable part of the intellectual equipment of every cultured person. Today the traditional place of mathematics in education is in grave danger. (Richard Courant and Herbert Robbins)

Acknowledgement

References

[1] Joseph Auslander, Review of "What is Mathematics, Really?" by Reuben Hersh, SIAM Review, 42(1) (2000), 138-143.

[2] Briane E. Blank, Review of "The Pleasures of Counting" by T. W. Körner, Notices Amer. Math. Soc. 45 (1998), 396-400.

[3] Douglas M. Campbell, John C. Higgins, Mathematics: People, Problems, Results (Volume 2, Taylor & Francis, 1984).

[4] Alain Connes, A view of Mathematics (38 pages), http://www.alainconnes.org/docs/maths.pdf

[5] J. J. O'Connor and E F Robertson, The MacTutor History of Mathematics archive (2000), http://www-history.mcs.st-and.ac.uk/

[6] Richard Courant and Herbert Robbins, What Is Mathematics? : An Elementary Approach to Ideas and Methods (1941); second edition, revised by Ian Stewart (Oxford University Press, 1996).

[7] Underwood Dudley, What Is Mathematics For? , Notices of AMS, Vol. 57, No. 5 (2010), 608-613.

[8] Paul Ernest, The Nature of Mathematics; https://www.academia.edu/3188583/ THE_NATURE_OF_MATHEMATICS_AND_TEACHING.

[9] Leonard Gillman, Review of "What Is Mathematics?", by Richard Courant and Herbert Robbins, revised by Ian Stewart , Amer. Math. Monthly, 105 (1998), 485-488.

[10] Arend Heyting, Intuitionistic Views on the Nature of Mathematics, Synthese, 27(1/2), On the Foundations of Mathematics, (May - Jun., 1974), pp. 79-91.

[11] Reuben Hersh, What is Mathematics, Really? (Oxford University Press, 1997).

[12] T. W. Körner, The Pleasures of Counting (Cambridge University Press, 1996).

[13] Kenneth Kunen, The Foundations of Mathematics (Studies in Logic: Mathematical Logic and Foundations) (College Publications, 2009).

[14] Robert H. Lewis, What is Mathematics? The Most Misunderstood Subject, http://www.fordham.edu/info/20603/what_math

[15] Ernst Snapper, "The three crises in mathematics: Logicism, intuitionism and formalism", Math. Magzine, 52(4) (1979), 207-216.

[16] Ernst Snapper, What is Mathematics?, Amer. Math. Monthly, 87(7) (1979), 551-557.

[17] Ian Stewart, Concepts of Modern Mathematics (Dover Publications, 1975).

[18] David Tall, The Transition to Formal Thinking in Mathematics, Mathematics Education Research Journal, 20(2) (2008), 5-24.

[19] Alan Weir, "Formalism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Spring 2015 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2015/entries/formalism-mathematics/>.

[20] Richard B. Wells, Mathematics and Mathematical Axioms (Ch. 23, 2006, 42 pages), http://www.mrc.uidaho.edu/~rwells/Critical%20Philosophy%20and%20Mind/Chapter%2023.pdf