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Total Szeged Index, Vertex-Edge Wiener Index and Edge Hyper-Wiener Index of Certain Special Molecular Graphs

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Abstract

In past years, topological indices are introduced to measure the characters of chemical molecules. Thus, the study of these topological indices has raised large attention in the field of chemical science, biology science and pharmaceutical science. In this paper, by virtue of molecular structure analysis, we determine the total Szeged index, vertex-edge Wiener index and edge hyper-Wiener index of several crucial molecular graphs, such as fan molecular graph, wheel molecular graph, gear fan molecular graph, gear wheel molecular graph, and their r-corona molecular graphs. The results achieved in our paper illustrate the promising application prospects for chemical engineering.

Keywords

Chemical Graph Theory, Total Szeged Index, Vertex-Edge Wiener Index, Edge Hyper-Wiener Index, Fan Molecular Graph, Wheel Molecular Graph, Gear Fan Molecular Graph, Gear Wheel Molecular Graph, *r*-Corona Molecular Graph

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1. Introduction

Wiener index, PI index and Zagreb indices are introduced to reflect certain structural features of organic molecules. Several papers contributed to determine the distance-based index of special molecular graphs (See Yan et al., [1-2], Gao et al., [3-4], Gao and Shi [5], Gao and Wang [6], Xi and Gao [7-8], Xi et al., [9], Gao et al., [10] for more detail).

Let $e=uv$ be an edge of the molecular graph G. The number of edges and vertices of G whose distance to the vertex u is smaller than the distance to the vertex v is denoted by $t_u(e)$. Analogously, $t_v(e)$ is the number of edges and vertices of G whose distance to the vertex v is smaller than the distance to the vertex u. The total Szeged index of G is defined as

$$Sz_T(G)=\sum_{e=uv} t_u(e)t_v(e).$$

Manuel et. al., [11] determined the total Szeged Index of C4-Nanotubes,C4-Nanotori and Dendrimer Nanostars. It is well know that

$$Sz_T(G)=Sz_e(G)+Sz_v(G)+2Sz_{ev}(G),$$

where $Sz_e(G)$ is edge-Szeged index, $Sz_v(G)$ is vertex-Szeged index, and $Sz_{ev}(G)$ is edge-vertex Szeged index.

Let u be a vertex in molecular graph G. For the edges $e=ab$ and $f=xy$, let

$$d'(u,e)=D'(e,u)=\min\{d(u,a),\mathrm{d}(u,b)\}$$

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and

$$D(e,f)=\min\{D'(e,x),D'(e,y)\}.$$

The vertex-edge Wiener index is introduced by Cevik and Maden [12] as follows:

$$W_{ev}(G)=\frac{1}{2}\sum_{f\in E}\sum_{v\in V}d'(v,f).$$

Similar as Wiener index and hyper-Wiener index, we define the edge hyper-Wiener index of molecular graph G as follows

$$WW_e(G)=\frac{1}{2}\sum_{\{e,f\}\subseteq E(G)}\{d(e,f)+d^2(e,f)\}.$$

Let P_n and C_n be path and cycle with n vertices. The molecular graph $F_n=\{v\}\vee P_n$ is called a fan molecular graph and the molecular graph $W_n=\{v\}\vee C_n$ is denoted as a wheel molecular graph. Molecular graph $I_r(G)$ is called r- crown molecular graph of G which splicing r hang edges for every vertex in G. By adding one vertex in every two adjacent vertices of the fan path P_n of fan molecular graph F_n, the resulting molecular graph is a subdivision molecular graph called gear fan molecular graph, denote as $\tilde{F}_n$. By adding one vertex in every two adjacent vertices of the wheel cycle C_n of wheel molecular graph W_n, The resulting molecular graph is a subdivision molecular graph, called gear wheel molecular graph, denoted as $\tilde{W}_n$.

As we know, fan molecular graph, wheel molecular graph, gear fan molecular graph, gear wheel molecular graph and r-corona molecular graph are common molecular structure in chemical. Hence, it inspires us to study the topological indices of these important chemical molecular structures. In this paper, we present the total Szeged index, vertex-edge Wiener index and edge hyper-Wiener index of $I_r(F_n)$, $I_r(W_n)$, $I_r(\tilde{F}_n)$ and $I_r(\tilde{W}_n)$. The rest parts of article are organized as follows: first, we compute the total Szeged index of above molecular graphs; then, the vertex-edge Wiener index of such chemical structures are presented; at last, we derive the edge hyper-Wiener index of these special molecular graphs.

2. Total Szeged Index

Theorem1. $Sz_T(I_r(F_n))$ = $r^2(8n^2+16n-18)+r(18n^2+14n-62)+(9n^2-32)$.

Proof. Let $P_n=v_1v_2\ldots v_n$ and the r hanging vertices of v_i be v_i^1, v_i^2,…, v_i^r $(1\le i\le n)$. Let v be a vertex in F_n beside P_n, and the r hanging vertices of v be v^1, v^2, …, v^r. Using the definition of total Szeged index, we have

$$\begin{aligned}Sz_T(I_r(F_n))&=\sum_{i=1}^{r}(t_v(vv^i)t_{v^i}(vv^i))+\sum_{i=1}^{n}(t_v(vv_i)t_{v_i}(vv_i))+\sum_{i=1}^{n-1}(t_{v_i}(v_iv_{i+1})t_{v_{i+1}}(v_iv_{i+1}))+\sum_{i=1}^{n}\sum_{j=1}^{r}(t_{v_i}(v_iv_i^j)t_{v_i^j}(v_iv_i^j))\\
&=r[(1+1)\times((2n+r+nr-2)+(r+n(r+1)))]+(2[((r+2)+(r+1))\times((2n+nr-2r-4)+(n-1)(r+1))]\\
&\quad+(n-2)[((r+2)+(r+1))\times((2n+nr-2r-5)+(n-2)(r+1))])\\
&\quad+(2[((2r+3)+2(r+1))\times((r+1)+(r+1))]+2[((2r+3)+2(r+1))\times((2r+2)+2(r+1))]\\
&\quad+(n-5)[((2r+3)+2(r+1))\times((2r+3)+2(r+1))])+nr[(1+1)\times((2n+r+nr-2)+(r+n(r+1)))]\\
&=r^2(8n^2+16n-18)+r(18n^2+14n-62)+(9n^2-32)\end{aligned}$$

Corollary 1. $Sz_T(F_n)=9n^2-32$.

Theorem 2. $Sz_T(I_r(W_n))=r^2(6n^2+24n-4)+r(12n^2+43n-28)+(5n^2+21n-18)$.

Proof. Let $C_n=v_1v_2\ldots v_n$ and v_i^1, v_i^2,…, v_i^r be the r hanging vertices of $v_i(1\le i\le n)$. Let v be a vertex in W_n beside C_n, and v^1, v^2, …, v^r be the r hanging vertices of v. We denote $v_nv_{n+1}=v_nv_1$. In view of the definition of total Szeged index, we infer

$$\begin{aligned}Sz_T(I_r(W_n))&=\sum_{i=1}^{r}(t_v(vv^i)t_{v^i}(vv^i))+\sum_{i=1}^{n}(t_v(vv_i)t_{v_i}(vv_i))+\sum_{i=1}^{n}(t_{v_i}(v_iv_{i+1})t_{v_{i+1}}(v_iv_{i+1}))+\sum_{i=1}^{n}\sum_{j=1}^{r}(t_{v_i}(v_iv_i^j)t_{v_i^j}(v_iv_i^j))\\
&=r[(1+1)\times((2n+r+nr-1)+(r+n(r+1)))]\\
&\quad+n[((r+2)+(r+1))\times((2n+nr-2r-5)+(n-2)(r+1))]\end{aligned}$$

$$+n[((2r+3)+2(1+r))\times((2r+3)+2(1+r))]+nr[(1+1)\times((2n+r+nr-1)+(r+n(r+1)))]$$

$$=r^2(6n^2+24n-4)+r(12n^2+43n-28)+(5n^2+21n-18).$$

Corollary 2. $Sz_T(W_n)=5n^2+21n-18$.

Theorem 3. $Sz_T(I_r(\tilde{F}_n))=r^2(88n^2-156n+96)+r(170n^2-390n+240)+(75n^2-204n+134)$.

Proof. Let $P_n=v_1v_2\ldots v_n$ and $v_{i,i+1}$ be the adding vertex between vi and vi+1. Let v_i^1, v_i^2,…, v_i^r be the r hanging vertices of vi (1 ≤ i ≤ n). Let $v_{i,i+1}^1$, $v_{i,i+1}^2$,…, $v_{i,i+1}^r$ be the r hanging vertices of $v_{i,i+1}$ (1 ≤ i ≤ n-1). Let v be a vertex in Fn beside P_n, and the r hanging vertices of v be v^1, v^2, …, v^r.

By virtue of the definition of total Szeged index, we yield

$$Sz_T(I_r(\tilde{F}_n))=\sum_{i=1}^{r}(t_v(vv^i)t_{v^i}(vv^i))+\sum_{i=1}^{n}(t_v(vv_i)t_{v_i}(vv_i))+\sum_{i=1}^{n}\sum_{j=1}^{r}(t_{v_i}(v_iv_i^j)t_{v_i^j}(v_iv_i^j))+\sum_{i=1}^{n-1}(t_{v_i}(v_iv_{i,i+1})t_{v_{i,i+1}}(v_iv_{i,i+1}))$$

$$+\sum_{i=1}^{n-1}(t_{v_{i,i+1}}(v_{i,i+1}v_{i+1})t_{v_{i+1}}(v_{i,i+1}v_{i+1}))+\sum_{i=1}^{n-1}\sum_{j=1}^{r}(t_{v_{i,i+1}}(v_{i,i+1}v^j_{i,i+1})t_{v^j_{i,i+1}}(v_{i,i+1}v^j_{i,i+1}))$$

$$=r[(1+1)\times((3n+2nr-3)+(r+(r+1)(2n-1))]$$

$$+(2[((2r+1)+2(r+1))\times((2nr+3n-2r-5)+(2n-2)(r+1))]$$

$$+(n-2)[((3r+2)+3(r+1))\times((2nr+3n-3r-7)+(2n-3)(r+1))])$$

$$+nr[(1+1)\times((3n+2nr-3)+(r+(r+1)(2n-1))]$$

$$+(n-1)[((2nr-3r+3n-7)+(2n-3)(r+1))\times((3r+2)+3(r+1))]$$

$$+(n-1)[((2nr-3r+3n-7)+(2n-3)(r+1))\times((3r+2)+3(r+1))]$$

$$+(n-1)r[(1+1)\times((3n+2nr-3)+(r+(r+1)(2n-1))]$$

$$=r^2(88n^2-156n+96)+r(170n^2-390n+240)+(75n^2-204n+134).$$

Corollary3. $Sz_T(\tilde{F}_n)=75n^2-204n+134$.

Theorem 4. $Sz_T(I_r(\tilde{W}_n))=r^2(88n^2-60n+2)+r(170n^2-169n+3)+(75n^2-95n-2)$.

Proof. Let $C_n=v_1v_2\ldots v_n$ and v be a vertex in W_n beside C_n, and $v_{i,i+1}$ be the adding vertex between v_i and v_{i+1}. Let v^1, v^2, …, v^r be the r hanging vertices of v and v_i^1, v_i^2,…, v_i^r be the r hanging vertices of vi (1 ≤ i ≤ n). Let $v_{n,n+1}=v_{1,n}$ and $v_{i,i+1}^1$, $v_{i,i+1}^2$,…, $v_{i,i+1}^r$ be the r hanging vertices of $v_{i,i+1}$ (1 ≤ i ≤ n). Let $v_{n.n+1}=v_{n.1}$, $v_{n+1}=v_1$. In view of the definition of total Szeged index, we deduce

$$Sz_T(I_r(\tilde{W}_n))=\sum_{i=1}^{r}(t_v(vv^i)t_{v^i}(vv^i))+\sum_{i=1}^{n}(t_v(vv_i)t_{v_i}(vv_i))+\sum_{i=1}^{n}\sum_{j=1}^{r}(t_{v_i}(v_iv_i^j)t_{v_i^j}(v_iv_i^j))+\sum_{i=1}^{n}(t_{v_i}(v_iv_{i,i+1})t_{v_{i,i+1}}(v_iv_{i,i+1}))$$

$$+\sum_{i=1}^{n}(t_{v_{i,i+1}}(v_{i,i+1}v_{i+1})t_{v_{i+1}}(v_{i,i+1}v_{i+1}))+\sum_{i=1}^{n}\sum_{j=1}^{r}(t_{v_{i,i+1}}(v_{i,i+1}v^j_{i,i+1})t_{v^j_{i,i+1}}(v_{i,i+1}v^j_{i,i+1}))$$

$$=r[(1+1)\times((3n+2nr+r-1)+(r+2n(r+1))]+n[((3r+2)+3(r+1))\times((2nr+3n-2r-5)+(2n-2)(r+1))]$$

$$+nr[(1+1)\times((3n+2nr+r-1)+(r+2n(r+1))]+n[((2nr-2r+3n-5)+(2n-2)(r+1))\times((3r+2)+3(r+1))]$$

$$+n[((2nr-2r+3n-5)+(2n-2)(r+1))\times((3r+2)+3(r+1))]+nr[(1+1)\times((3n+2nr+r-1)+(r+2n(r+1))]$$

$$=r^2(88n^2-60n+2)+r(170n^2-169n+3)+(75n^2-95n-2)\ .$$

Corollary 4. $Sz_T(\tilde{W}_n)=75n^2-95n-2$.

3. Vertex-Edge Wiener Index

The notations and terminologies for particular molecular graphs presented in this section can refer to section 2.

Theorem 5. $W_{ev}(I_r(F_n))=r^2(\frac{3}{2}n^2-\frac{1}{2}n+\frac{3}{2})+r(\frac{7}{2}n^2-\frac{9}{2}n+2)+(\frac{3}{2}n^2-4n+2)$.

Proof. Using the definition of vertex-edge Wiener index, we have

$$W_{ev}(I_r(F_n))=\frac{1}{2}\{\sum_{i=1}^{r}d'(v,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{r}d'(v^j,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n}d'(v_j,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,vv^i)+\sum_{i=1}^{n}d'(v,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{r}d'(v^j,vv_i)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_j,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,vv_i)+\sum_{i=1}^{n-1}d'(v,v_iv_{i+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{r}d'(v^j,v_iv_{i+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n}d'(v_j,v_iv_{i+1})$$

$$+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_iv_{i+1})+\sum_{i=1}^{n}\sum_{t=1}^{r}d'(v,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{r}d'(v^j,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}d'(v_j,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_iv_i^t)\}$$

$$=\frac{1}{2}\{0+(r^2-r)+rn+2nr^2+0+rn+(n^2-n)+r(2n^2-n)+(n-1)+r(2n-2)+(2n^2-8n+5)+r(3n^2-9n+5)+rn$$

$$+2nr^2+r(2n^2-4n+2)+r^2(3n^2-5n+2)\}$$

$$=r^2(\frac{3}{2}n^2-\frac{1}{2}n+\frac{3}{2})+r(\frac{7}{2}n^2-\frac{9}{2}n+2)+(\frac{3}{2}n^2-4n+2)\ .$$

Corollary 5. $W_{ev}(F_n)=\frac{3}{2}n^2-4n+2$.

Theorem 6. $W_{ev}(I_r(W_n))=r^2(\frac{3}{2}n^2-\frac{1}{2}n+\frac{1}{2})+r(4n^2-\frac{9}{2}n-\frac{1}{2})+(\frac{3}{2}n^2-3n)$.

Proof. In view of the definition of vertex-edge Wiener index, we infer

$$W_{ev}(I_r(W_n))=\frac{1}{2}\{\sum_{i=1}^{r}d'(v,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{r}d'(v^j,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n}d'(v_j,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,vv^i)+\sum_{i=1}^{n}d'(v,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{r}d'(v^j,vv_i)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_j,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,vv_i)+\sum_{i=1}^{n}d'(v,v_iv_{i+1})+\sum_{i=1}^{n}\sum_{j=1}^{r}d'(v^j,v_iv_{i+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_j,v_iv_{i+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_iv_{i+1})+\sum_{i=1}^{n}\sum_{t=1}^{r}d'(v,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{r}d'(v^j,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}d'(v_j,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_iv_i^t)\}$$

$$=\frac{1}{2}\{0+(r^2-r)+rn+2nr^2+0+rn+(n^2-n)+r(3n^2-4n)+n+2nr+(2n^2-6n)+r(3n^2-6n)+rn+2nr^2$$

$$+r(2n^2-4n)+r^2(3n^2-5n)\}$$

$$=r^2(\frac{3}{2}n^2-\frac{1}{2}n+\frac{1}{2})+r(4n^2-\frac{9}{2}n-\frac{1}{2})+(\frac{3}{2}n^2-3n)\ .$$

Corollary 6. $W_{ev}(W_n)=\frac{3}{2}n^2-3n$.

Theorem 7. $W_{ev}(I_r(\tilde{F}_n))=r^2(8n^2-14n+\frac{21}{2})+r(\frac{31}{2}n^2-\frac{67}{2}n+\frac{45}{2})+(\frac{13}{2}n^2-\frac{33}{2}n+13)$.

Proof. By virtue of the definition of vertex-edge Wiener index, we yield

$$\begin{aligned}
W_{ev}(I_r(\tilde{F}_n))=&\frac{1}{2}\{\sum_{i=1}^{r}d'(v,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{r}d'(v^j,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n}d'(v_j,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n-1}d'(v_{j,j+1},vv^i)\\
&+\sum_{i=1}^{r}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d'(v_{j,j+1}^k,vv^i)+\sum_{i=1}^{n}d'(v,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{r}d'(v^j,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_j,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,vv_i)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n-1}d'(v_{j,j+1},vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d'(v_{j,j+1}^k,vv_i)+\sum_{i=1}^{n-1}d'(v,v_iv_{i,i+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{r}d'(v^j,v_iv_{i,i+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n}d'(v_j,v_iv_{i,i+1})\\
&+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_iv_{i,i+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}d'(v_{j,j+1},v_iv_{i,i+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d'(v_{j,j+1}^k,v_iv_{i,i+1})+\sum_{i=1}^{n-1}d'(v,v_{i,i+1}v_{i+1})\\
&+\sum_{i=1}^{n-1}\sum_{j=1}^{r}d'(v^j,v_{i,i+1}v_{i+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n}d'(v_j,v_{i,i+1}v_{i+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_{i,i+1}v_{i+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}d'(v_{j,j+1},v_{i,i+1}v_{i+1})\\
&+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d'(v_{j,j+1}^k,v_{i,i+1}v_{i+1})+\sum_{i=1}^{n}\sum_{t=1}^{r}d'(v,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{r}d'(v^j,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}d'(v_j,v_iv_i^t)\\
&+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n-1}d'(v_{j,j+1},v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d'(v_{j,j+1}^k,v_iv_i^t)+\sum_{i=1}^{n-1}\sum_{t=1}^{r}d'(v,v_{i,i+1}v_{i,i+1}^t)\\
&+\sum_{i=1}^{n-1}\sum_{t=1}^{r}\sum_{j=1}^{r}d'(v^j,v_{i,i+1}v_{i,i+1}^t)+\sum_{i=1}^{n-1}\sum_{t=1}^{r}\sum_{j=1}^{n}d'(v_j,v_{i,i+1}v_{i,i+1}^t)+\sum_{i=1}^{n-1}\sum_{t=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_{i,i+1}v_{i,i+1}^t)+\sum_{i=1}^{n-1}\sum_{t=1}^{r}\sum_{j=1}^{n-1}d'(v_{j,j+1},v_{i,i+1}v_{i,i+1}^t)\\
&+\sum_{i=1}^{n-1}\sum_{t=1}^{r}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d'(v_{j,j+1}^k,v_{i,i+1}v_{i,i+1}^t)\}\\
=&\frac{1}{2}\{0+(r^2-r)+rn+2nr^2+r(2n-2)+r^2(3n-3)+0+rn+(n^2-n)+r(2n^2-n)+(2n^2-4n+2)\\
&+r(3n^2-5n+2)+(n-1)+r(2n-2)+(2n^2-3n+1)+r(3n^2-6n+1)+(3n^2-12n+12)+r(4n^2-14n+13)\\
&+(n-1)+r(2n-2)+(2n^2-3n+1)+r(3n^2-6n+1)+(3n^2-12n+12)+r(4n^2-14n+13)+rn+2nr^2\\
&+r(2n^2-2n)+r^2(3n^2-3n)+r(3n^2-7n+4)+r^2(4n^2-8n+4)+r(2n-2)+r^2(3n-3)+r(3n^2-7n+4)\\
&+r^2(4n^2-8n+4)+r(4n^2-16n+16)+r^2(5n^2-19n+18)\}\\
=&r^2(8n^2-14n+\frac{21}{2})+r(\frac{31}{2}n^2-\frac{67}{2}n+\frac{45}{2})+(\frac{13}{2}n^2-\frac{33}{2}n+13).
\end{aligned}$$

Corollary7. $W_{ev}(\tilde{F}_n)=\frac{13}{2}n^2-\frac{33}{2}n+13$.

Theorem 8. $W_{ev}(I_r(\tilde{W}_n))=r^2(8n^2-4n+\frac{1}{2})+r(\frac{31}{2}n^2-15n-\frac{1}{2})+(\frac{13}{2}n^2-\frac{19}{2}n)$.

Proof. In view of the definition of vertex-edge Wiener index, we deduce

$$W_{ev}(I_r(\tilde{W}_n))=\frac{1}{2}\{\sum_{i=1}^{r}d'(v,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{r}d'(v^j,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n}d'(v_j,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,vv^i)+\sum_{i=1}^{r}\sum_{j=1}^{n}d'(v_{j,j+1},vv^i)$$

$$+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_{j,j+1}^k,vv^i)+\sum_{i=1}^{n}d'(v,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{r}d'(v^j,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_j,vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,vv_i)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_{j,j+1},vv_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_{j,j+1}^k,vv_i)+\sum_{i=1}^{n}d'(v,v_iv_{i,i+1})+\sum_{i=1}^{n}\sum_{j=1}^{r}d'(v^j,v_iv_{i,i+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_j,v_iv_{i,i+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_iv_{i,i+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_{j,j+1},v_iv_{i,i+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_{j,j+1}^k,v_iv_{i,i+1})+\sum_{i=1}^{n}d'(v,v_{i,i+1}v_{i+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{r}d'(v^j,v_{i,i+1}v_{i+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_j,v_{i,i+1}v_{i+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_{i,i+1}v_{i+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d'(v_{j,j+1},v_{i,i+1}v_{i+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_{j,j+1}^k,v_{i,i+1}v_{i+1})+\sum_{i=1}^{n}\sum_{t=1}^{r}d'(v,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{r}d'(v^j,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}d'(v_j,v_iv_i^t)$$

$$+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}d'(v_{j,j+1},v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_{j,j+1}^k,v_iv_i^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}d'(v,v_{i,i+1}v_{i,i+1}^t)$$

$$+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{r}d'(v^j,v_{i,i+1}v_{i,i+1}^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}d'(v_j,v_{i,i+1}v_{i,i+1}^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_j^k,v_{i,i+1}v_{i,i+1}^t)$$

$$+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}d'(v_{j,j+1},v_{i,i+1}v_{i,i+1}^t)+\sum_{i=1}^{n}\sum_{t=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d'(v_{j,j+1}^k,v_{i,i+1}v_{i,i+1}^t)\}$$

$$=\frac{1}{2}\{0+(r^2-r)+rn+2nr^2+2nr+3nr^2+0+rn+(n^2-n)+r(2n^2-n)+(2n^2-2n)+r(3n^2-2n)+n+2nr$$

$$+(2n^2-3n)+r(3n^2-3n)+(3n^2-6n)+r(4n^2-6n)+n+2nr+(2n^2-3n)+r(3n^2-3n)+(3n^2-6n)$$

$$+r(4n^2-6n)+rn+2nr^2+r(2n^2-2n)+r^2(3n^2-3n)+r(3n^2-4n)+r^2(4n^2-4n)+2nr+3nr^2+r(3n^2-4n)$$

$$+r^2(4n^2-4n)+r(4n^2-8n)\quad+\quad r^2(5n^2-9n)\}$$

$$=r^2(8n^2-4n+\frac{1}{2})+r(\frac{31}{2}n^2-15n-\frac{1}{2})+(\frac{13}{2}n^2-\frac{19}{2}n).$$

Corollary 8. $W_{ev}(\tilde{W}_n)=\frac{13}{2}n^2-\frac{19}{2}n$.

4. Edge Hyper-Wiener Index

The notations and terminologies for special molecular graphs used in this section can refer to section 2.

Theorem 9. $WW_e(I_r(F_n))=r^2(3n^2-\frac{5}{2}n+\frac{7}{2})+r(9n^2-21n+18)+(\frac{13}{2}n^2-\frac{49}{2}n+29)$.

Proof. By the definition of edge hyper-Wiener index, we have

$$WW_e(I_r(F_n))=\frac{1}{2}\{\{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}d(vv^i,vv^j)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d(v_iv_i^j,v_iv_i^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d(v_iv_i^k,v_jv_j^t)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv^i,v_jv_j^k)$$

$$+\sum_{i=1}^{r}\sum_{j=1}^{n}d(vv^i,vv_j)+\sum_{i=1}^{r}\sum_{j=1}^{n-1}d(vv^i,v_jv_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv_i,v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{k=1}^{r}d(v_iv_{i+1},v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d(vv_i,vv_j)$$

$$+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}d(v_iv_{i+1},v_jv_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n-1}d(vv_i,v_jv_{j+1})\}+\{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}d^2(vv^i,vv^j)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d^2(v_iv_i^j,v_iv_i^k)$$

$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d^2(v_iv_i^k,v_jv_j^t)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv^i,v_jv_j^k)+\sum_{i=1}^{r}\sum_{j=1}^{n}d^2(vv^i,vv_j)+\sum_{i=1}^{r}\sum_{j=1}^{n-1}d^2(vv^i,v_jv_{j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv_i,v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(v_iv_{i+1},v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d^2(vv_i,vv_j)+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}d^2(v_iv_{i+1},v_jv_{j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n-1}d^2(vv_i,v_jv_{j+1})\}\}$$

$$=\frac{1}{2}\{\{\frac{r(r-1)}{2}+\frac{nr(r-1)}{2}+r^2(\frac{3}{2}n^2-\frac{5n}{2}+1)+2nr^2+rn+2r(n-1)+r(2n^2-n)+r(3n^2-9n+8)$$

$$+\frac{n(n-1)}{2}+(\frac{3}{2}n^2-\frac{15}{2}n+10)+(2n^2-4n+2)\}+\{\frac{r(r-1)}{2}+\frac{nr(r-1)}{2}+r^2(\frac{9}{2}n^2-\frac{19n}{2}+5)+4nr^2$$

$$+rn+4r(n-1)+r(4n^2-3n)+r(9n^2-35n+36)+\frac{n(n-1)}{2}+(\frac{9}{2}n^2-\frac{53}{2}n+40)+(4n^2-10n+6)\}\}$$

$$=\frac{1}{2}\{\{r^2(\frac{3}{2}n^2+\frac{3}{2})+r(5n^2-\frac{15}{2}n+\frac{11}{2})+(4n^2-12n+12)\}+\{r^2(\frac{9}{2}n^2-5n+\frac{11}{2})+r(13n^2-\frac{69}{2}n+\frac{61}{2})$$

$$+(9n^2-37n+46)\}\}$$

$$=r^2(3n^2-\frac{5}{2}n+\frac{7}{2})+r(9n^2-21n+18)+(\frac{13}{2}n^2-\frac{49}{2}n+29).$$

Corollary 9. $WW_e(F_n)=\frac{13}{2}n^2-\frac{49}{2}n+29$.

Theorem 10. $WW_e(I_r(W_n))=r^2(3n^2-\frac{5}{2}n+\frac{1}{2})+r(9n^2-\frac{29}{2}n-\frac{1}{2})+(\frac{13}{2}n^2-\frac{31}{2}n)$.

Proof. By the definition of edge hyper-Wiener index, we have

$$WW_e(I_r(W_n))=\frac{1}{2}\{\{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}d(vv^i,vv^j)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d(v_iv_i^j,v_iv_i^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d(v_iv_i^k,v_jv_j^t)\ \sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv^i,v_jv_j^k)$$

$$+\sum_{i=1}^{r}\sum_{j=1}^{n}d(vv^i,vv_j)+\sum_{i=1}^{r}\sum_{j=1}^{n}d(vv^i,v_jv_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv_i,v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(v_iv_{i+1},v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d(vv_i,vv_j)$$

$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d(v_iv_{i+1},v_jv_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d(vv_i,v_jv_{j+1})\}+\{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}d^2(vv^i,vv^j)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d^2(v_iv_i^j,v_iv_i^k)$$

$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d^2(v_iv_i^k,v_jv_j^t)\ \sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv^i,v_jv_j^k)+\sum_{i=1}^{r}\sum_{j=1}^{n}d^2(vv^i,vv_j)+\sum_{i=1}^{r}\sum_{j=1}^{n}d^2(vv^i,v_jv_{j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv_i,v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(v_iv_{i+1},v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d^2(vv_i,vv_j)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d^2(v_iv_{i+1},v_jv_{j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}d^2(vv_i,v_jv_{j+1})\}\}$$

$$=\frac{1}{2}\{\{\frac{r(r-1)}{2}+\frac{nr(r-1)}{2}+r^2(\frac{3}{2}n^2-\frac{5}{2}n)+2nr^2+rn+2rn+r(2n^2-n)+r(3n^2-6n)+\frac{n(n-1)}{2}$$

$$+(\frac{3}{2}n^2-\frac{9}{2}n)+(2n^2-2n)\}+\{\frac{r(r-1)}{2}+\frac{nr(r-1)}{2}+r^2(\frac{9}{2}n^2-\frac{19}{2}n)+4nr^2+rn+4rn+r(4n^2-3n)$$

$$+r(9n^2-26n)+\frac{n(n-1)}{2}+(\frac{9}{2}n^2-\frac{35}{2}n)+(4n^2-6n)\}\}\}$$

$$=\frac{1}{2}\{\{r^2(\frac{3}{2}n^2+\frac{1}{2})+r(5n^2-\frac{9}{2}n-\frac{1}{2})+(4n^2-7n)\}+\{r^2(\frac{9}{2}n^2-5n+\frac{1}{2})+r(13n^2-\frac{49}{2}n-\frac{1}{2})+(9n^2-24n)\}\}$$

$$=r^2(3n^2-\frac{5}{2}n+\frac{1}{2})+r(9n^2-\frac{29}{2}n-\frac{1}{2})+(\frac{13}{2}n^2-\frac{31}{2}n).$$

Corollary 10. $WW_e(W_n)=\frac{13}{2}n^2-\frac{31}{2}n$.

Theorem 11. $WW_e(I_r(\tilde{F}_n))=r^2(\frac{41}{2}n^2-53n+\frac{83}{2})+r(41n^2-144n+98)+(\frac{37}{2}n^2-\frac{115}{2}n+45)$.

Proof. By virtue of the definition of edge hyper-Wiener index, we get

$$WW_e(I_r(\tilde{F}_n))=\frac{1}{2}\{\{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}d(vv^i,vv^j)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d(v_iv_i^j,v_iv_i^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d(v_iv_i^k,v_jv_j^t)+\sum_{i=1}^{n-1}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d(v_{i,i+1}v_{i,i+1}^j,v_{i,i+1}v_{i,i+1}^k)$$

$$+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=1}^{r}\sum_{t=1}^{r}d(v_{i,i+1}v_{i,i+1}^k,v_{j,j+1}v_{j,j+1}^t)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv^i,v_jv_j^k)+\sum_{i=1}^{r}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d(vv^i,v_{j,j+1}v_{j,j+1}^k)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=1}^{r}\sum_{t=1}^{r}d(v_iv_i^k,v_{j,j+1}v_{j,j+1}^t)+\sum_{i=1}^{r}\sum_{j=1}^{n}d(vv^i,vv_j)+\sum_{i=1}^{r}\sum_{j=1}^{n-1}d(vv^i,v_jv_{j,j+1})+\sum_{i=1}^{r}\sum_{j=1}^{n-1}d(vv^i,v_{j,j+1}v_{j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv_i,v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{k=1}^{r}d(v_iv_{i,i+1},v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{k=1}^{r}d(v_{i,i+1}v_{i+1},v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d(vv_i,v_{j,j+1}v_{j,j+1}^k)$$

$$+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d(v_iv_{i,i+1},v_{j,j+1}v_{j,j+1}^k)+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d(v_{i,i+1}v_{i+1},v_{j,j+1}v_{j,j+1}^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d(vv_i,vv_j)+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}d(v_iv_{i,i+1},v_jv_{j.j+1})$$

$$+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}d(v_{i,i+1}v_{i+1},v_{j.j+1}v_{j+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}d(v_iv_{i,i+1},v_{j.j+1}v_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n-1}d(vv_i,v_jv_{j,j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n-1}d(vv_i,v_{j,j+1}v_{j+1})\}$$

$$+\{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}d^2(vv^i,vv^j)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d^2(v_iv_i^j,v_iv_i^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d^2(v_iv_i^k,v_jv_j^t)+\sum_{i=1}^{n-1}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d^2(v_{i,i+1}v_{i,i+1}^j,v_{i,i+1}v_{i,i+1}^k)$$

$$+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=1}^{r}\sum_{t=1}^{r}d^2(v_{i,i+1}v_{i,i+1}^k,v_{j,j+1}v_{j,j+1}^t)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv^i,v_jv_j^k)+\sum_{i=1}^{r}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d^2(vv^i,v_{j,j+1}v_{j,j+1}^k)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=1}^{r}\sum_{t=1}^{r}d^2(v_iv_i^k,v_{j,j+1}v_{j,j+1}^t)+\sum_{i=1}^{r}\sum_{j=1}^{n}d^2(vv^i,vv_j)+\sum_{i=1}^{r}\sum_{j=1}^{n-1}d^2(vv^i,v_jv_{j,j+1})+\sum_{i=1}^{r}\sum_{j=1}^{n-1}d^2(vv^i,v_{j,j+1}v_{j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv_i,v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(v_iv_{i,i+1},v_jv_j^k)+\sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(v_{i,i+1}v_{i+1},v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d^2(vv_i,v_{j,j+1}v_{j,j+1}^k)$$

$$+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d^2(v_iv_{i,i+1},v_{j,j+1}v^k_{j,j+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{r}d^2(v_{i,i+1}v_{i+1},v_{j,j+1}v^k_{j,j+1})+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d^2(vv_i,vv_j)+$$
$$\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}d^2(v_iv_{i,i+1},v_jv_{j.j+1})$$

$$+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}d^2(v_{i,i+1}v_{i+1},v_{j.j+1}v_{j+1})+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}d^2(v_iv_{i,i+1},v_{j.j+1}v_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n-1}d^2(vv_i,v_jv_{j,j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n-1}d^2(vv_i,v_{j,j+1}v_{j+1})\}\}$$

$$=\frac{1}{2}\{\{\frac{r(r-1)}{2}+\frac{nr(r-1)}{2}+r^2(\frac{3}{2}n^2-\frac{5}{2}n+1)+\frac{(n-1)r(r-1)}{2}+r^2(\frac{5}{2}n^2-\frac{19}{2}n+9)+2r^2n+3r^2(n-1)$$

$$+4r^2(n^2-2n+1)+rn+2r(n-1)+2r(n-1)+r(2n^2-n)+r(3n^2-6n+3)+r(3n^2-6n+3)+r(3n^2-5n+2)$$

$$+r(4n^2-12n+13)+r(4n^2-12n+13)+\frac{n(n-1)}{2}+(\frac{3}{2}n^2-\frac{11}{2}n+5)+(\frac{3}{2}n^2-\frac{11}{2}n+5)+(3n^2-12n+9)$$

$$+(2n^2-3n)+(2n^2-3n)\}+\{\frac{r(r-1)}{2}+\frac{nr(r-1)}{2}+r^2(\frac{9}{2}n^2-\frac{9}{2}n)+\frac{(n-1)r(r-1)}{2}+r^2(\frac{25}{2}n^2-\frac{107}{2}n+57)$$

$$+4r^2n+9r^2(n-1)+r^2(16n^2-40n+24)+rn+4r(n-1)+4r(n-1)+r(4n^2-3n)+r(9n^2-22n+13)$$

$$+r(9n^2-22n+13)+r(9n^2-19n+10)+r(16n^2-96n+69)+r(16n^2-96n+69)+\frac{n(n-1)}{2}+(\frac{9}{2}n^2-\frac{37}{2}n+19)$$

$$+(\frac{9}{2}n^2-\frac{37}{2}n+19)+(9n^2-34n+33)+(4n^2-7n)+(4n^2-7n)\}\}$$

$$=\frac{1}{2}\{\{r^2(8n^2-14n+11)+r(19n^2-38n+30)+(\frac{21}{2}n^2-\frac{59}{2}n+19)\}+\{r^2(33n^2-92n+72)$$

$$+r(63n^2-250n+166)+(\frac{53}{2}n^2-\frac{171}{2}n+71)\}\}$$

$$=r^2(\frac{41}{2}n^2-53n+\frac{83}{2})+r(41n^2-144n+98)+(\frac{37}{2}n^2-\frac{115}{2}n+45).$$

Corollary11. $WW_e(\tilde{F}_n)=\frac{37}{2}n^2-\frac{115}{2}n+45$.

Theorem 12. $WW_e(I_r(\tilde{W}_n))=r^2(\frac{41}{2}n^2-\frac{47}{2}n+\frac{1}{2})+r(41n^2-58n-\frac{1}{2})+(\frac{37}{2}n^2-\frac{63}{2}n)$.

Proof. In view of the definition of edge hyper-Wiener index, we deduce

$$WW_e(I_r(\tilde{W}_n))=\frac{1}{2}\{\{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}d(vv^i,vv^j)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d(v_iv_i^j,v_iv_i^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d(v_iv_i^k,v_jv_j^t)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d(v_{i,i+1}v^j_{i,i+1},v_{i,i+1}v^k_{i,i+1})$$

$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d(v_{i,i+1}v^k_{i,i+1},v_{j,j+1}v^t_{j,j+1})+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv^i,v_jv_j^k)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv^i,v_{j,j+1}v^k_{j,j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d(v_iv_i^k,v_{j,j+1}v^t_{j,j+1})+\sum_{i=1}^{r}\sum_{j=1}^{n}d(vv^i,vv_j)+\sum_{i=1}^{r}\sum_{j=1}^{n}d(vv^i,v_jv_{j,j+1})+\sum_{i=1}^{r}\sum_{j=1}^{n}d(vv^i,v_{j,j+1}v_{j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv_i,v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(v_iv_{i,i+1},v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(v_{i,i+1}v_{i+1},v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(vv_i,v_{j,j+1}v^k_{j,j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(v_iv_{i,i+1},v_{j,j+1}v^k_{j,j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d(v_{i,i+1}v_{i+1},v_{j,j+1}v^k_{j,j+1})+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d(vv_i,vv_j)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d(v_iv_{i,i+1},v_jv_{j,j+1})$$

$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d(v_{i,i+1}v_{i+1},v_{j,j+1}v_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d(v_iv_{i,i+1},v_{j,j+1}v_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d(vv_i,v_jv_{j,j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d(vv_i,v_{j,j+1}v_{j+1})\}$$

$$+\{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}d^2(vv^i,vv^j)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d^2(v_iv_i^j,v_iv_i^k)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d^2(v_iv_i^k,v_jv_j^t)+\sum_{i=1}^{n}\sum_{j=1}^{r-1}\sum_{k=j+1}^{r}d^2(v_{i,i+1}v^j_{i,i+1},v_{i,i+1}v^k_{i,i+1})$$

$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d^2(v_{i,i+1}v^k_{i,i+1},v_{j,j+1}v^t_{j,j+1})+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv^i,v_jv_j^k)+\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv^i,v_{j,j+1}v^k_{j,j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}\sum_{t=1}^{r}d^2(v_iv_i^k,v_{j,j+1}v^t_{j,j+1})+\sum_{i=1}^{r}\sum_{j=1}^{n}d^2(vv^i,vv_j)+\sum_{i=1}^{r}\sum_{j=1}^{n}d^2(vv^i,v_jv_{j,j+1})+\sum_{i=1}^{r}\sum_{j=1}^{n}d^2(vv^i,v_{j,j+1}v_{j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv_i,v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(v_iv_{i,i+1},v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(v_{i,i+1}v_{i+1},v_jv_j^k)+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(vv_i,v_{j,j+1}v^k_{j,j+1})$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(v_iv_{i,i+1},v_{j,j+1}v^k_{j,j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{r}d^2(v_{i,i+1}v_{i+1},v_{j,j+1}v^k_{j,j+1})+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d^2(vv_i,vv_j)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d^2(v_iv_{i,i+1},v_jv_{j,j+1})$$

$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d^2(v_{i,i+1}v_{i+1},v_{j,j+1}v_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d^2(v_iv_{i,i+1},v_{j,j+1}v_{j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d^2(vv_i,v_jv_{j,j+1})+\sum_{i=1}^{n}\sum_{j=1}^{n}d^2(vv_i,v_{j,j+1}v_{j+1})\}\}$$

$$=\frac{1}{2}\{\{\frac{r(r-1)}{2}+\frac{nr(r-1)}{2}+r^2(\frac{3}{2}n^2-\frac{3}{2}n)+\frac{nr(r-1)}{2}+r^2(\frac{5}{2}n^2-\frac{9}{2}n)+2r^2n+3r^2n+4r^2(n^2-n)+rn+2rn$$

$$+2rn+r(2n^2-n)+r(3n^2-3n)+r(3n^2-3n)+r(3n^2-2n)+r(4n^2-6n)+r(4n^2-6n)+\frac{n(n-1)}{2}+(\frac{3}{2}n^2-\frac{5}{2}n)$$

$$+(\frac{3}{2}n^2-\frac{5}{2}n)+(3n^2-4n)+(2n^2-n)+(2n^2-n)\}+\{\frac{r(r-1)}{2}+\frac{nr(r-1)}{2}+r^2(\frac{9}{2}n^2-\frac{9}{2}n)+\frac{nr(r-1)}{2}$$

$$+r^2(\frac{25}{2}n^2-\frac{57}{2}n)+4r^2n+9r^2n+r^2(16n^2-24n)+rn+4rn+4rn+r(4n^2-3n)+r(9n^2-13n)+r(9n^2-13n)$$

$$+r(9n^2-10n)+r(16n^2-34n)+r(16n^2-34n)+\frac{n(n-1)}{2}+(\frac{9}{2}n^2-\frac{19}{2}n)+(\frac{9}{2}n^2-\frac{19}{2}n)+(9n^2-26n)$$

$$+(4n^2-3n)+(4n^2-3n)\}\}$$

$$=\frac{1}{2}\{r^2(8n^2-4n+\frac{1}{2})+r(19n^2-17n-\frac{1}{2})+(\frac{21}{2}n^2-\frac{23}{2}n)\}+\{r^2(33n^2-43n+\frac{1}{2})+r(63n^2-99n-\frac{1}{2})$$

$$+(\frac{53}{2}n^2-\frac{103}{2}n)\}\}$$

$$=r^2(\frac{41}{2}n^2-\frac{47}{2}n+\frac{1}{2})+r(41n^2-58n-\frac{1}{2})+(\frac{37}{2}n^2-\frac{63}{2}n).$$

Corollary12. $WW_e(\tilde{W}_n)=\frac{37}{2}n^2-\frac{63}{2}n$.

5. Future Work

Let $e_G(v)$ be the eccentricity of a vertex v in a connected molecular graph G, n=$|V(G)|$ be a number of vertices, and m=$|E(G)|$ be a number of edges. Cevik and Maden [12] defined following average class of molecular graph invariants:

- Average eccentricity: $avec(G) = \frac{1}{n}\sum_{v\in V(G)} e_G(v) = \frac{1}{n}\sum_{v\in V(G)} \max_{u\in V(G)} d(u,v)$.
- average edge eccentricity: $avec_e(G) = \frac{1}{m}\sum_{g\in E(G)} \max_{f\in E(G)} D(g,f)$.
- average vertex-edge eccentricity: $avec_{ev}(G) = \frac{1}{n}\sum_{v\in V(G)} \max_{f\in E(G)} d'(v,f)$.
- weighted average eccentricity: $avec_+(G) = \frac{1}{n}\sum_{v\in V(G)} \deg(v)e_G(v)$.

Hence, these average indices of special structural molecules graphs can be consider as subjects for future research.

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