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# On Natural Partial Orders of IC-Abundant Semigroups

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## Abstract

In this paper, we will investigate the natural partial orders on IC-abundant semigroups. After giving some properties and characterizations of natural partial orders on abundant semigroups, we consider IC-abundant semigroups. We prove that an IC-abundant semigroup is locally ample if and only if the natural partial order on the semigroup is compatible with the multiplication.

## Keywords

Abundant Semigroup, Natural Partial Order, IC-Abundant Semigroup, Locally Ample

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## 1. Introduction

The concepts of the natural partial orders on a regular semigroup were introduced by Nambooripad [3] in 1980. As a generalization of regular semigroups in the range of abundant semigroups, El-Qallali and Fountain [ 1 ] introduced abundant semigroups. After that, various classes of abundant semigroups are researched (see, [2, 6-14] ). In 1987, Lawson defined natural partial orders on abundant semigroups, and extend Nambooripad's results. The relation $R^*$ is defined on a semigroup $S$ by the rule that $aR^*b$ if and only if the elements $a,b$ of $S$ are related by Green's relation $R$ in some oversemigroup of $S$. The relation $L^*$ is defined dually. In this paper, we shall study the natural partial orders on IC-abundant semigroups by using the notion of majorization of the relations $R^*$ and $L^*$. Some properties and constructions on IC-abundant semigroups will be described in terms of its natural partial order. We shall proceed as follows: section 2 provides some known results. In section 3, we give some characterizations of the natural partial orders on abundant semigroups. The last section we consider the natural partial orders on IC-abundant semigroups.

## 2. Preliminaries

Throughout this paper we shall use the notions and notations of [2,4-5]. Here we provide some known results used repeatedly in the sequel. At first, we recall some basic facts about the relation $L^*$ and $R^*$.

Lemma *II*. 1 [1] Let $S$ be a semigroup and $a,b\in S$ . Then the following statements are equivalent:

(1) $aL^*b$ ( $aR^*b$ );

(2) for all $x,y\in S^1$ $ax=ay(xa=ya)$ if and only if $bx=by(xb=yb)$.

As an easy but useful consequence of Lemma II.1, we have

*Corollary II.1* [1] Let $S$ be a semigroup and $a,e=e^2\in S$ .Then the following statements are equivalent:

(1) $aL^*e$ ( $aR^*e$ );

(2) $a=ae(a=ea)$ and for all $x,y\in S^1$ $ax=ay(xa=ya)$

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implies $bx = by (xb = yb)$.

Evidently, $L^*$ is a right congruence while $R^*$ is a left congruence. In an arbitrary semigroup, we have $L \subseteq L^*$ and $R \subseteq R^*$. But for regular elements $a, b$, we get $aL^*b(aR^*b)$ if and only if $aLb(aRb)$. For convenience, we denote by $a^+$ [ $a^*$ ] a typical idempotent $R^*$ related [ $L^*$ related ] to $a$. $L_a^*$ ( $R_a^*$ ) denotes the $L^*$ class ( $R^*$ class ) containing $a$. And $E(T)$ denotes the set of idempotents of $T$; $\mathrm{Reg}(T)$ denotes the set of regular elements of $T$. We denote by $V(a)$ the set of all inverses of $a$.

A semigroup $S$ is called *abundant* if and only if each $L^*$ class and each $R^*$ class contains at least one idempotent. An abundant semigroup $S$ is called *quasi-adequate* if its set of idempotents constitutes a subsemigroup (i.e., its set of idempotents is a band). Moreover, a quasi-adequate semigroup is called *adequate* if its bands of idempotents is a semilattice (i.e., the idempotents commute). An abundant semigroup $S$ is called *ample,* if for all $e^2 = e, a \in S$, $ae = (ae)^+ a$ and $ea = a(ea)^*$. Follwing[1], an abundant semigroup $S$ is called *idempotent-connected*, for short, *IC*, provided for each $a \in S$ and for some $a^+, a^*$, there exists a bijection $\theta : < a^+ > \to < a^* >$ such that $xa = a(x\theta)$ for all $x \in < a^+ >$, where $< a^+ >$ [resp. $< a^* >$ ] is the subsemigroup of $S$ generated by the set $\{y \in E(S) : y = ya^+ = a^+ y\}$ [resp. $\{y \in E(S) : y = ya^* = a^* y\}$ ]. In fact, an ample semigroup is an IC-adequate semigroup and vice versa. A semigroup $S$ is called a *locally P-semigroup* if for all $e^2 = e \in S$, $eSe$ is a P-semigroup. An equivalence relation $\rho$ on $S$ is called $\rho$ *-unipotent* if each $\rho$ -class of $S$ contains exactly one idempotent. Evidently, an adequate semigroup is both $L^*$- and $R^*$- unipotent. It is well known that $L$ - unipotent [ resp. $R$ - unipotent ] regular semigroup is an orthdox semigroup whose band of idempotents is a right [ resp. left ] regular band ( a band $B$ is left [resp. right] regular band if for $x, y \in B$, $xy = xyx$ [resp. $xy = yxy$ ][ see,4]).

*Definition II.1*[5] Let $S$ be an abundant semigroup. We define three relations on $S$, as follows:

(1) $a \le_r b \Leftrightarrow R_a^* \le R_b^*$, and there exists an idempotent $f \in R_a^*$ such that $a = fb$;

(2) $a \le_l b \Leftrightarrow L_a^* \le L_b^*$, and there exists an idempotent $e \in L_a^*$ such that $a = be$;

(3) $\le = \le_r \cap \le_l$, i.e. $a \le b \Leftrightarrow$ there exist idempotents $e, f$ such that $a = eb = bf$.

*Lemma II. 2* [5] Let $S$ be an abundant semigroup and $a, b \in S$. Then $a \le_r b$ if and only if there exists $e \in R_a^* \cap E(S), f \in R_b^* \cap E(S)$ such that $a = eb$ and $e \le f$.

*Lemma II.3* [5] Let $S$ be an abundant semigroup. Then $S$ is IC if and only if $\le_r = \le = \le_l$.

*Definition II.2* [4] Let $\rho$ be an equivalence relation on semigroup $S$, $T$ be a subset of $S$. $T$ is called to satisfy $\rho$-majorization if for any $a, b, c \in T, a \ge b, a \ge c$ and $b\rho c$ implies that $b = c$.

## 3. Properties and Characterizations

The aim of this section is to introduce the natural partial on abundant semigroups, and to give some properties and characterizations of the natural partials on such semigroups.

*PropositionIII.1* Let $S$ be an abundant semigroup and $a, b \in S$. Then $a \le b$ if and only if there exist $x, y \in S^1$ such that $a = xb = by$ and $xa = a$.

*Proof.* We only prove the sufficiency part. To see this, let $x, y \in S^1$ such that $a = xb = by$ and $xa = a$. Then $a = xa = xby = ay$. Hence, by Corollary II.1, $a^* = a^* y$ and $a^+ = xa^+$. Furthermore, we get $ya^*, a^+ x \in E(S)$. Thus $a = a^+ a = a^+(xb) = (a^+ x)b = aa^* = (by)a^* = b(ya^*)$. This means that $a \le b$.

*LemmaIII.1* Let $S$ be an abundant semigroup. If $S$ satisfy $L^*$-majorization, then for any $e \in E(S)$, $eSe$ is $L^*$- unipotent.

*Proof.* Let $f, g \in E(eSe)$. Then $e \ge f$ and $e \ge g$. If $f$ $L^*(eSe)$ $g$, then $f$ $L^*(S)$ $g$. Since $S$ satisfy $L^*$-majorization, we have $f = g$. So, $eSe$ is $L^*$- unipotent.

*LemmaIII.2* Let $S$ be an abundant semigroup. If $S$ is $L^*$-unipotent, then $S$ satisfy $L^*$- majorization.

*Proof.* Let $a, b, c \in S$ such that $a \ge b, a \ge c$ and $bL^*c$. By the dual argument of Lemma II.2, there exist $e_1 \in E(L_b^*), e_2 \in E(L_c^*)$ such that $b = ae_1, c = ae_2$. Hence, $e_1 L^* b L^* c L^* e_2$. But, $S$ is $L^*$- unipotent , we have $e_1 = e_2$. So, $b = ae_1 = ae_2 = c$, that is, $S$ satisfy $L^*$- majorization.

*Proposition III.2* Let $S$ be an abundant semigroup. Then the following statements are equivalent:

(1) for any $e \in E(S)$, $eSe$ is $\mathcal{L}^*$- unipotent;

(2) for any $e \in E(S)$, $eSe$ satisfy $\mathcal{L}^*$- majorization;

(3) $E(S)$ satisfy $\mathcal{L}$- majorization;

(4) $Reg(S)$ satisfy $\mathcal{L}$- majorization.

*Proof.* (1)$\Rightarrow$(2) It follows immediately from Lemma III.2.

(2)$\Rightarrow$(3) Suppose that (2) holds. Let $e, f, g \in E(S)$ and $e \geq f, e \geq g$, $f\ \mathcal{L}\ g$. Then $f, g \in E(eSe)$ and $f\ \mathcal{L}^*(eSe)\ g$. Hence, $f = g$, and so $E(S)$ satisfy $\mathcal{L}$- majorization.

(3)$\Rightarrow$(4) Assume that $E(S)$ satisfy $\mathcal{L}$- majorization. Let $a, b, c \in Reg(S)$ and $a \geq b, a \geq c$, $b\mathcal{L}(S)c$. Then there exist $e, f \in E(S)$ such that $b = ea = af$. On the other hand, since $a$ is regular, we have $x \in V(a)$ such that $(xb)^2 = xeaxaf = xeaf = xb \in E(S)$. Again, since

$xb = (xax)b = (xa)(xb) = xea = (xea)xa = (xb)(xa),$

we have that $xb \leq xa$. Notice that $b = af = axaf = a(xb)$, we get that $b\mathcal{L}(S)xb$. Similarly, we can prove that $xc \in E(S), c = a(xc), xc \leq xa$ and $c\mathcal{L}(S)xc$. Hence, $xb\mathcal{L}(S)xc$. By the hypothesis, $E(S)$ satisfy $\mathcal{L}$- majorization, we have that $xb = xc$. Therefore, $b = a(xb) = a(xc) = c$. That is, $Reg(S)$ satisfy $\mathcal{L}$- majorization.

(4)$\Rightarrow$(1) Assume that $Reg(S)$ satisfy $\mathcal{L}$- majorization. Let $f, g \in E(eSe)$ and $f\ \mathcal{L}^*(eSe)\ g$. Hence, $e \geq f, e \geq g$ and $f\ \mathcal{L}(S)\ g$. By (4), we have $f = g$, and so $eSe$ is $\mathcal{L}^*$- unipotent.

*Theorem III. 1* Let $S$ be an abundant semigroup satisfying the regularity condition. Then the following statements are equivalent:

(1) $Reg(S)$ is compatible with respect to $\leq$;

(2) $S$ is a locally adequate semigroup;

(3) for any $e \in E(eSe)$, $eSe$ satisfies both $\mathcal{L}^*$- and $\mathcal{R}^*$-unipotent;

(4) for any $e \in E(eSe)$, $eSe$ satisfies both $\mathcal{L}^*$- and $\mathcal{R}^*$- majorization;

(5) $E(S)$ satisfies both $\mathcal{L}$- and $\mathcal{R}$- majorization;

(6) $Reg(S)$ satisfies both $\mathcal{L}$- and $\mathcal{R}$- majorization.

*Proof.* (1)$\Rightarrow$(2) Suppose that $Reg(S)$ is compatible with respect to $\leq$. Then $Reg(S)$ is a locally inverse semigroup (see, [ 3 ]). Noticing that for any $e \in E(S)$, $eSe$ is an abundant semigroup, which implies that $E(eReg(S)e)$ is a semilattice and $Reg(eSe) = eReg(S)e$. We observe that $E(eSe)$ is a semilattice (since $eReg(S)e$ is a inverse semigroup ). Thus, $eSe$ is an adequate semigroup, that is, $S$ is a locally adequate semigroup.

(2)$\Rightarrow$(3) This is clear.

(3)$\Rightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(6) Follow from Proposition III.2.

(6)$\Rightarrow$(1) Suppose that $Reg(S)$ satisfies both $\mathcal{L}$- and $\mathcal{R}$- majorization. By Proposition III.2, we have $eSe$ satisfies both $\mathcal{L}^*$- and $\mathcal{R}^*$-unipotent. But, $Reg(eSe) = eReg(S)e$, we have that $Reg(eSe)$ is a regular semigroup satisfying both $\mathcal{L}^*$- and $\mathcal{R}^*$-unipotent. Furthermore, $Reg(eSe)$ is an inverse semigroup. Therefore, $Reg(S)$ is a locally inverse semigroup, and so $Reg(S)$ is compatible with respect to $\leq$ (see,[ 3 ]).

## 4. Natural Partial Orders on IC-Abundant Semigroups

In this section, we will consider the natural partial orders on IC-abundant semigroups.

Theorem IV.1 Let $S$ be an IC-abundant semigroup. Then the following statements are equivalent:

(1) $S$ is right compatible with respect to $\leq$;

(2)for any $e, f, g \in E(S), e \geq f$ and $e \geq g \Rightarrow fg = gfg$;

(3)for any $e \in E(S)$, $eSe$ satisfies both $\mathcal{L}^*$- majorization and the regularity condition;

(4) for any $e \in E(S)$, $eSe$ satisfies both $\mathcal{L}^*$- unipotent and the regularity condition.

Proof. (1)$\Rightarrow$(2) Let $e, f, g \in E(S), e \geq f$ and $e \geq g$. By (1), we have that $eg \geq fg$. Hence, $g \geq fg$. By Proposition 2.7 of [5 ], we have $fg \in E(S)$. Therefore, $fg = gfg$.

(2)$\Rightarrow$(3) Let $f, g \in E(eSe)$. Then $e \geq f$ and $e \geq g$. By (2), we have that $fg = gfg$.

Hence, $(fg)^2 = fgfg = f(fg) = fg \in E(S)$, which implies that $eSe$ satisfies the regularity condition.

Next , we show that $eSe$ satisfies $\mathcal{L}^*$- majorization.

Let $a,b,c\in eSe$ such that $a\geq b$, $a\geq c$ and $bL(S)c$. By the duality of Lemma II.2, there exist $e_1\in E(L_b^*)\bigcap eSe, e_2\in E(L_c^*)\bigcap eSe$ such that $b=ae_1, c=ae_2$. Obviously, $e\geq e_1$, $e\geq e_2$ and $e_1L(eSe)e_2$. Hence, $e_1e_2=e_2e_1e_2$, so that $e_1=e_1e_2=e_2e_1e_2=e_2e_2=e_2$. Thus $b=ae_1=ae_2=c$. Therefore, $eSe$ satisfies $L^*$- majorization.

(3)$\Rightarrow$ (4) It follows from Proposition III.2.

(4)$\Rightarrow$ (1) Suppose that (4) holds. Then $\mathrm{Re}\,g(eSe)$ is an $L^*$-unipotent semigroup. Hence,

$E(eSe)$ is a right regular band. Let $a,b,c\in S$ and $a\leq b$. By Lemma II.2, for $f\in E(R_b^*)$,

there exists $e\in E(R_a^*)$ such that $e\leq f$ and $a=eb$. Hence, $ac=ebc$. Since $fbc=bc$, by LemmaII.1, we have $f(bc)^+=(bc)^+\in E(R_{bc}^*)$. It is easy to see that $(bc)^+f\in E(R_{bc}^*)$. Thus $e(bc)^+f\in E(fSf)$ since $e,(bc)^+f\in E(fSf)$. Take $x\in V(e(bc)^+f)\bigcap fSf$. We have

$$xe(bc)^+f,(bc)^+fxe(bc)^+f\in E(fSf)\text{ and}$$
$$xe(bc)^+fL(fSf)(bc)^+fxe(bc)^+f,$$

that is,

$$xe(bc)^+fL^*(fSf)(bc)^+fxe(bc)^+f.$$

Since $E(fSf)$ is a right regular band, by assumpation, we have

$$xe(bc)^+f=(bc)^+fxe(bc)^+f.$$

Multiplying the prior formula on the right by $x$, we obtain that $x=(bc)^+fx$. Hence, $x=xe(bc)^+fx=xex$, and so, $xe,e(bc)^+fxe\in E(fSf)$. Notice that $xeL(fSf)e(bc)^+fxe$, by the fact that $fSf$ is $L^*$- unipotent, we have that $xe=e(bc)^+fxe$. If we multiply this equality on the right by $(bc)^+f$, we can obtain that $xe(bc)^+f=e(bc)^+f$. Hence, $e(bc)^+f=e(bc)^+fxe(bc)^+f=e(bc)^+fe(bc)^+f$. That is, $e(bc)^+f\in E(fSf)$. Note that $(bc)^+f,(bc)^+fe(bc)^+f\in E(fSf)$, $e(bc)^+fL^*(fSf)(bc)^+fe(bc)^+f$, by (4), we observe $(bc)^+fe(bc)^+f=e(bc)^+f$. Hence, $e(bc)^+f\leq(bc)^+f$. Since $ac=ebc=e(bc)^+fbc$, by Lemma II.2, we have $ac\leq_r bc$. Again, since $S$ is an IC-abundant semigroup, we get that $ac\leq bc$. The proof is completed.

*Theorem IV.2* Let $S$ be an IC-abundant semigroup. Then the following statements are equivalent:

(1) $S$ is compatible with respect to $\leq$;

(2)for any $e,f,g\in E(S), e\geq f$ and $e\geq g\Rightarrow fg=gf$;

(3) $S$ is a locally ample semigroup;

(4) for any $e\in E(S)$, $eSe$ satisfies both $L^*$- and $R^*$-majorization and the regularity condition.

*Proof.* It follows from Theorem IV.1 and its dual.

## 5. Conclusions

In this paper, we investigate the natural partial orders on IC-abundant semigroups and give some properties and characterizations of natural partial orders on abundant semigroups by using the notion of majorization. We generalize and strengthen the results of Fountain on abundant semigroups.

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