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A Special Class of Tensor Product Surfaces with Harmonic Gauss Map

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Abstract

In this paper, we stat a necessary and sufficient condition for Gauss map of the tensor product of planar unite circle and a special smooth curve in En to be harmonic. In this way, we construct two orthonormal basis for the tangent space and the normal space of the resulting tensor product surface. As a direct consequence of these basis, we also get a result about shape operators of this surface.

Keywords

Tensor Product, Gauss Map, Harmonic Map

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1. Introduction

The tensor product of two immersions of a given Riemannian manifold was introduced by Chen in the late 1970's ([3]). This notion is a generalization of the quadratic representation of a submanifold.

In a special case, a tensor product surface is obtained by taking the tensor product of two curves. A number of properties such as minimality and totally reality are studied about tensor product of two planar curves ([7]). Moreover, minimal and pseudo- minimal tensor product of Lorentzian planar curve and an Euclidean planar curve is considered by Mihai ([8]).

Gauss map is one of the topics in differential geometry. On the other hand, harmonic functions have very useful properties in advanced mathematics. So, we study the tensor product surfaces of two curves that have harmonic Gauss map.

2. Preliminaries

In this section, we recall some standard definitions and results from Riemannian geometry. Let M be an n-dimensional manifold, E^m be an m-dimensional Euclidean space and $\varphi: M \to E^m$ be an isometric immersion, as well as $\bar{\nabla}$ the Levi-Civita connection of E^m and ∇ the induced connection on M from E^m. We denote the second fundamental form of M in E^m by II, normal connection in the normal bundle of M by $\tilde{\nabla}$ and the shape operator in the direction of normal vector field n by A_n. It is well known that the two later notions are related to each other by

$$< II(X, Y), n > = < A_n, Y > \qquad (1)$$

where X and Y are tangent vector fields to M. For an n-dimensional submanifold M in E^m, the mean curvature vector $\vec{H}$ is given by

$$\vec{H} = \frac{1}{n} trace\ II$$

If $\vec{H} \equiv 0$, then the submanifold is said to be minimal. A submanifold is called totally geodesic if $II \equiv 0$. Furthermore, the Gaussian and Weingarten formula are given, respectively, by

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$$\overline{\nabla}_X Y = \nabla_X Y + II(X,Y) \quad (2)$$

$$\overline{\nabla}_X n = -A_n X + \widetilde{\nabla}_X n. \quad (3)$$

Using above notations, we have the following Ricci equation,

$$< [A_{n_1}, A_{n_2}]X, Y >= 01 \quad (4)$$

for tangent vector fields X, Y and normal vector fields n_1 and n_2.

Let $G(n,m)$ be the Grassmannian consisting of all oriented n-planes through the origin of E^m. For an isometric immersion Γ:M→E^m, the Gauss map $\upsilon: M \to G(n,m)$ of Γ is a smooth map which carries $p \in M$ into the oriented n-plane in E^m, which obtained from the parallel translation of T_pM, the tangent space of M at p in E^m. We known that $G(n,m)$ canonically imbedded in $\Lambda^n E^m$, the vector space obtained by the exterior product of n vectors in E^m. We can assume $\Lambda^n E^m$ as Euclidean space E^N where $N = \binom{m}{n}$, so the Gauss map at $p \in M$ can be written as $\upsilon(p) = (e_1 \Lambda e_2 \Lambda \dots \Lambda e_n)(\text{p})$, where $\{e_1, e_2, \dots, e_n, e_{n+1}, \dots, e_m\}$ is an adapted local orthonormal frame field in E^m such that $e_1, e_2, \dots, e_n$ are tangent to M and $e_{n+1}, \dots, e_m$ are normal to M. If $C^\infty(M)$ be the set of real smooth functions on M, then the Laplacian of $f \in C^\infty(M)$ is defined by

$$\Delta f = -\textstyle\sum_i (\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} f - \overline{\nabla}_{\nabla_{e_i} e_i} f) \quad (5)$$

Note that in this context, smooth can be replaced by C^2.

3. A Special Tensor Product Surface in E^{2n}

Let $\boldsymbol{c_1}: \boldsymbol{R} \to \boldsymbol{E^2}$ be the unit planar circle centered at the origin with parameterization $\boldsymbol{c_1(s) = (\cos s, \sin s)}$ and $\boldsymbol{c_2}: \boldsymbol{R} \to$be a unit speed smooth curve in $\boldsymbol{E^n}$ with parametrization $\boldsymbol{c_2(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t))}$. **Here, we consider** $\boldsymbol{\alpha_1(t) \neq 0}$, $\boldsymbol{\alpha'_1(t) \neq 0}$ for every $\boldsymbol{t \in R}$ (index 1 can be replaced by $\boldsymbol{i, i = 2, \dots, n}$). The tensor product surface $\boldsymbol{M}$ of two curves $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ is given by,

$$f = c_1 \otimes c_2: R^2 \to E^{2n}$$

$$\boldsymbol{f(s,t) = (\alpha_1(t)\cos s, \alpha_2(t)\cos s, .., \alpha_n(t)\cos s,}$$
$$\boldsymbol{\alpha_1(t)\sin s, \alpha_2(t)\sin s, .., \alpha_n(t)\sin s)}$$

Assume that $f(s,t) = c_1(s) \otimes c_2(t)$ defines an isometric immersion of R^2 into E^{2n}. Let prime denote derivative with respect to t. It is easily seen that

$$e_1 = \frac{1}{||c_2||}(-\alpha_1(t)\sin s, \dots, -\alpha_n(t)\sin s, \alpha_1(t)\cos s, \dots,$$
$$\alpha_n(t)\cos s)$$

and

$$e_2 = (\alpha'_1(t)\cos s, \dots, \alpha'_n(t)\cos s, \alpha'_1(t)\sin s, \dots, \alpha'_n(t)\sin s)$$

form an orthonormal frame for tangent space of M. Moreover, an orthonormal basis normal to M is given by

$$e_{2i+1} = T_i(\alpha_{2i}(t)\sin s, 0, \dots, 0, \underbrace{-\alpha_1(t)\sin s}_{2i-th}, 0, \dots,$$
$$0, \underbrace{-\alpha_{2i}(t)\cos s}_{(n+1)-th}, 0, \dots, 0, \underbrace{\alpha_1(t)\cos s}_{(n+2i)-th}, 0, \dots, 0)$$

$$e_{2j} = S_j(\alpha'_{2j-1}(t)\cos s, 0, \dots, 0, \underbrace{-\alpha'_1(t)\cos s}_{(2j-1)-th}, 0, \dots,$$
$$0, \underbrace{\alpha'_{2j-1}(t)\sin s}_{(n+1)-th}, 0, \dots, 0, \underbrace{-\alpha'_1(t)\sin s}_{(n+2j-1)-th}, 0, \dots, 0)$$

where $\boldsymbol{1 \le 2i+1 \le 2n-1}$, $\boldsymbol{2 \le 2j \le 2n}$,

$$T_i = \frac{1}{\sqrt{\alpha_1^2(t) + \alpha_{2i}^2(t)}} \text{ and } S_j = \frac{1}{\sqrt{\alpha_1^2(t) + \alpha_{2j-1}^2(t)}}.$$

If we use the following abbreviation,

$$A = A(t) = -\frac{\sum_{i=1}^n \alpha_i(t)\alpha'_i(t)}{||c_2||^2}$$

and for $1 \le i \le n-1$

$$B_i = B_i(t) = -\frac{\alpha_1(t)\alpha'_1(t) + \alpha_{i+1}(t)\alpha'_{i+1}(t)}{\alpha_1^2(t) + \alpha_{i+1}^2(t)}$$

$$C_i = C_i(t) = -\frac{\alpha'_1(t)\alpha'''_1(t) + \alpha'_{i+1}(t)\alpha'''_{i+1}(t)}{\alpha'^2_1(t) + \alpha'^2_{i+1}(t)}$$

$$D_i = D_i(t) = \frac{\alpha_1(t)\alpha''_{i+1}(t) - \alpha'_1(t)\alpha_{i+1}(t)}{\sqrt{\alpha_1^2(t) + \alpha_{i+1}^2(t)}}$$

$$E_i = E_i(t) = \frac{\alpha''_1(t)\alpha'_{i+1}(t) - \alpha'_1(t)\alpha''_{i+1}(t)}{\sqrt{\alpha'^2_1(t) + \alpha'^2_{i+1}(t)}}$$

$$F_i = F_i(t) = \frac{\alpha'_1(t)\alpha_{i+1}(t) - \alpha_1(t)\alpha'_{i+1}(t)}{\sqrt{\alpha'^2_1(t) + \alpha'^2_{i+1}(t)}}$$

$$L_i = L_i(t) = \sqrt{\frac{\alpha_1^2(t) + \alpha_{i+1}^2(t)}{\alpha'^2_1(t) + \alpha'^2_{i+1}(t)}}$$

also for $\boldsymbol{2 \le i < j \le n}$, we us

$$G_{ij} = G_{ij}(t) = \frac{\alpha_i(t)\alpha'_j(t)}{\sqrt{\left(\alpha_1^2(t) + \alpha_i^2(t)\right)\left(\alpha'^2_1(t) + \alpha'^2_j(t)\right)}}$$

$$\bar{G}_{ij} = \bar{G}_{ij}(t) = \frac{-\alpha'_i(t)\alpha_j(t)}{\sqrt{\left(\alpha'^2_1(t) + \alpha'^2_i(t)\right)(\alpha^2_1(t) + \alpha^2_j(t))}}$$

$$H_{ij} = H_{ij}(t) = \frac{B_{j-1}\alpha_i(t)\alpha_j(t) + \alpha_i(t)\alpha'_j(t)}{\sqrt{\left(\alpha^2_1(t) + \alpha^2_i(t)\right)(\alpha^2_1(t) + \alpha^2_j(t))}}$$

$$\bar{H}_{ij} = \bar{H}_{ij}(t) = \frac{B_{i-1}\alpha_i(t)\alpha_j(t) + \alpha'_i(t)\alpha_j(t)}{\sqrt{\left(\alpha^2_1(t) + \alpha^2_i(t)\right)\left(\alpha^2_1(t) + \alpha^2_j(t)\right)}}$$

$$K_{ij} = K_{ij}(t) = \frac{C_{j-1}\alpha'_i(t)\alpha'_j(t) + \alpha'_i(t)\alpha''_j(t)}{\sqrt{\left(\alpha'^2_1(t) + \alpha'^2_i(t)\right)(\alpha'^2_1(t) + \alpha'^2_j(t))}}$$

$$\bar{K}_{ij} = \bar{K}_{ij}(t) = \frac{C_{i-1}\alpha''_i(t)\alpha'_j(t) + \alpha'_i(t)\alpha''_j(t)}{\sqrt{\left(\alpha'^2_1(t) + \alpha'^2_i(t)\right)(\alpha'^2_1(t) + \alpha'^2_j(t))}}$$

we get,

$$\bar{\nabla}_{e_1}e_1 = \left(A\|c_2\|\right)e_2 + \sum_{i=1}^{n-1}\left(\frac{F_i}{\|C_2\|}\right)e_{2i+2}$$

$$\bar{\nabla}_{e_2}e_1 = \sum_{i=1}^{n-1}\left(\frac{D_i}{\|C_2\|}\right)e_{2i+1} \qquad (6)$$

$$\bar{\nabla}_{e_1}e_2 = \left(A\|c_2\|\right)e_1 + \sum_{i=1}^{n-1}(D_i)e_{2i+1}$$

$$\bar{\nabla}_{e_2}e_2 = \sum_{i=1}^{n-1}(E_i)e_{2i+2}$$

and for $2 \le i \le n$ we have,

$$\bar{\nabla}_{e_1}e_{2i-1} = (-D_{i-1})e_2 + \sum_{j=1, j\neq i}^{n-1}(V_i^j)e_{2j+2} - (B_iL_i)e_{2i+2}$$

$$\bar{\nabla}_{e_2}e_{2i-1} = \left(-\frac{D_{i-1}}{\|c_2\|}\right)e_2 + \sum_{j=1,j\neq i}^{n-2}(W_i^j)e_{2j+3} \qquad (7)$$

$$\bar{\nabla}_{e_1}e_{2i} = \left(-\frac{F_{i-1}}{\|c_2\|}\right)e_1 + \sum_{j=1, j\neq i}^{n-1}(P_i^j)e_{2j+1} + (B_iL_i)e_{2i+1}$$

$$\bar{\nabla}_{e_2}e_{2i} = (-E_{i-1})e_1 + \sum_{j=1, j\neq i}^{n-2}(Q_i^j)e_{2j}$$

where V_i^j and P_i^j's are in $\{\pm G_{kl}\,,\ \pm\bar{G}_{kl}\}_{2\le k<l\le n}$, $W_i^j \in \{\pm H_{kl}\,,\ \pm\bar{H}_{kl}\}_{2\le k<l\le n}$ and $Q_i^j \in \{\pm K_{kl}\,,\ \pm\bar{K}_{kl}\}_{2\le k<l\le n}$.

One immediate result that follows from (3) and (7) is following Corollary.

Corollary. Let $f = c_1 \otimes c_2$ be the tensor product surface of the circle $c_1(s) = (\cos s\,, \sin s)$ and unit speed smooth curve $c_2(t) = \left(\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)\right)$, then for $1 \le i \le n-1$, , we have

$$A_{e_{2i-1}} = \begin{bmatrix} 0 & D_i \\ \frac{D_i}{\|C_2\|} & 0 \end{bmatrix}, \quad A_{e_{2i}} = \begin{bmatrix} \frac{F_i}{\|C_2\|} & 0 \\ 0 & E_i \end{bmatrix} \qquad (8)$$

The following Theorem provides us a necessary and sufficient condition for our special tensor product surface to have harmonic Gauss map

Theorem. *Let M be a tensor product surface of the circle* $c_1(s) = (\cos s\,, \sin s)$ *and unit speed smooth curve* $c_2(t) = \left(\alpha_1(t), \alpha_2(t), ., \alpha_n(t)\right)$. *The Gauss map of M is harmonic, if and only if M is a part of plane.*

Proof. If we use (5), (6) and (7), then a direct computation shows that the Laplacian of the Gauss map ν is given by

$$-\Delta v = \sum_{i=1}^{2}\sum_{i}(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}\vartheta - \bar{\nabla}_{\nabla_{e_i}e_i}\nu)$$

$$= -\left[\sum_{I=1}^{n-1}\frac{F_i^2}{\|c_2\|^2} + \left(\sum_{I=1}^{n-1}D_i^2\right)\left(\frac{1}{\|c_2\|^2}+1\right) + \sum_{I=1}^{n-1}E_i^2\right](e_1\Lambda e_2)$$

$$+\bar{T}_1(e_1\Lambda e_4) + \bar{T}_2(e_1\Lambda e_6) + \cdots + \bar{T}_{n-1}(e_1\Lambda e_{2n}) + \bar{S}_1(e_2\Lambda e_3)$$

$$+\bar{S}_2(e_2\Lambda e_5) + \cdots + \bar{S}_{n-1}(e_2\Lambda e_{2n-1})$$

$$+\left[\frac{2D_1}{\|c_2\|}(E_1 - F_1)\right](e_3\Lambda e_4) + \left[\frac{2D_1}{\|c_2\|}(E_2 - F_2)\right](e_3\Lambda e_6) + \cdots + \left[\frac{2D_1}{\|c_2\|}(E_{n-1} - F_{n-1})\right](e_3\Lambda e_{2n}) \qquad (9)$$

$$+\left[\frac{2D_2}{\|c_2\|}(F_1 - E_1)\right](e_4\Lambda e_5) + \left[\frac{2D_3}{\|c_2\|}(F_1 - E_1)\right](e_4\Lambda e_7) + \cdots$$

$$+\left[\frac{2D_{n-1}}{\|c_2\|}(F_1 - E_1)\right](e_4\Lambda e_{2n-1}) + \cdots$$

$$+\left[\frac{2D_{n-1}}{\|c_2\|}(F_{n-2} - E_{n-2})\right](e_{2n-2}\Lambda e_{2n-1})$$

$$+\left[\frac{2D_{n-2}}{\|c_2\|}(E_{n-1} - F_{n-1})\right](e_{2n-1}\Lambda e_{2n})$$

where for $1 \le i \le n-1$,

$$\bar{T}_i = F_iA + \sum_{j=1, j\neq i}^{n-1}D_iR_i + E_iA\|c_2\| + E'_i - B_iD_iL_i + \sum_{j=1, j\neq i}^{n-1}E_jM_j\,,$$

$$\bar{S}_j = \sum_{j=1, j\neq i}^{n-1}\frac{F_j\bar{R}_j}{\|c_2\|} + D_iA\left(\|c_2\| - 1\right) - \sum_{j=1, j\neq i}^{n-1}\frac{D_jN_j}{\|c_2\|} - \left(\frac{D_i}{\|c_2\|}\right)' + \frac{B_jN_j}{\|c_2\|}$$

$R_i, \bar{R}_i \in \{\pm G_{kl}, \pm\bar{G}_{kl}\}_{2\le k<l\le n}$, $M_i \in \{\pm K_{kl}, \pm\bar{K}_{kl}\}_{2\le k<l\le n}$ and $N_i \in \{\pm H_{kl}, \pm\bar{H}_{kl}\}_{2\le k<l\le n}$.

In definitions of $\bar{T}_i$ and $\bar{S}_j$, prime means the derivation respect to t. If the Gauss map of M is harmonic, i.e. $\Delta\upsilon = 0$, then (9) implies that

$$\sum_{l=1}^{n-1}\frac{F_i^2}{||c_2||^2} + \left(\sum_{l=1}^{n-1} D_i^2\right)\left(\frac{1}{||c_2||^2}+1\right) + \sum_{l=1}^{n-1} E_i^2 = 0 \quad (10)$$

$$\bar{T}_1 = 0, \bar{T}_2 = 0 \ \ldots, \quad \bar{T}_{n-1} = 0\,,$$

$$\bar{S}_1 = 0,\ \bar{S}_2 = 0\ , \ldots, \bar{S}_{n-1} = 0,\ \frac{2D_1}{||c_2||}(E_1 - F_1) = 0,$$

$$\frac{2D_1}{||c_2||}(E_2 - F_2) = 0, \quad \frac{2D_1}{||c_2||}(E_{n-1} - F_{n-1}) = 0$$

$$\frac{2D_2}{||c_2||}(F_1 - E_1) = 0, \quad \frac{2D_3}{||c_2||}(F_1 - E_1) = 0, \ldots,$$

$$\frac{2D_{n-1}}{||c_2||}(F_1 - E_1) = 0, \qquad \ldots, \quad \frac{2D_{n-1}}{||c_2||}(F_{n-2} - E_{n-2}) = 0$$

$$\frac{2D_{n-2}}{||c_2||}(E_{n-1} - F_{n-1}) = 0.$$

Since all terms on the right-hand side of the first equation in (10) are nonnegative, hence we have

$$\mathrm{D_i} \equiv \mathrm{E_i} \equiv \mathrm{F_i} \equiv 0$$

for $1 \le \mathrm{i} \le \mathrm{n}$. This result and (8), show that M is a totally geodesic surface in $\mathrm{R}^{2\mathrm{n}}$ and so M is a part of a plane. The converse is obvious.

4. Conclusion

The main conclusion of this paper is a planar surface in even dimensional Euclidean space can be obtained from tensor product of unit circle with a unit speed curve in an Euclidean surface of half dimension.

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