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# Static Spherically Symmetric Anisotropic Solutions of Einstein's Equations in General Relativity

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## Abstract

For the highly nonlinearity conditions, it is so much difficult to obtain exact solutions of Einstein's equations. Many authors have been working on the investigation of exact solutions of Einstein's equations. There are a fair number of static spherically symmetric exact solutions of Einstein's equations. One of these solutions, Schwarzschild uniform density solution is unphysical. In this paper, we have demonstrated that field equations for static spherically anisotropic spacetimes can be reduced to Riccati type differential equations. Moreover, we have presented three new techniques for finding static spherically symmetric anisotropic solutions of Einstein's equations. Using one of these techniques, a class of new solutions is generated. The solution is realistic and physically acceptable.

## Keywords

Radial Pressure, Tangential Pressure, Energy Density, Isotropic Solution, Anisotropic Solution, Hydrostatic Equilibrium, Polytrope Index

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## 1. Introduction

Einstein's equations describing static spherically symmetric perfect fluid distribution provide a system of three independent differential equations [1-8] for four unknown functions, namely the pressure function p(r), the energy density $\rho(r)$ and two metric functions. Therefore, additional information in the form of an equation of state or specification of one of the two metric functions is needed in order to solve the system. Evidently the solution generating techniques mentioned above needs one solution generating function, called the source function. On the other hand Einstein's equations describing static spherically symmetric anisotropic fluid distribution provide a system of three independent differential equations [9, 10] for five unknown functions, namely the radial pressure $p_r(r)$, tangential pressure $p_t(r)$, energy density $\rho(r)$ and two metric functions. Thus generation of such solutions needs two input functions. However instead of specifying two input functions anisotropic solutions can be generated by specifying one input function and an additional ansatz e.g. conformally flat condition, a known isotropic/ anisotropic solution [11-15] etc.

In Section-2, we have derived the field equations governing static spherically symmetric an anisotropic fluid distribution in the form needed for our future purpose. In Section-3, techniques for generating static spherically symmetric anisotropic solutions are reviewed. The techniques are illustrated with the help of examples. In Section-4, new techniques for generating static spherically

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symmetric anisotropic solutions are presented. In Section-5, a class of new solutions is generated using the new techniques. In Section-6, we discuss on the properties of the solution. Finally in Section-7, some concluding remarks are given.

## 2. Einstein's Field Equations for Static Anisotropic Fluid Spheres

The line element for space-time describing static spherically symmetric geometry in curvature coordinates can be written as

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

For the metric (1) Einstein's tensor $G_{\alpha\beta}$ is given by

$$G_{00} = \frac{e^{2\phi}}{r^2}\left(1 - e^{-2\Lambda} + 2re^{-2\Lambda}\Lambda'\right)$$

$$G_{rr} = -\frac{e^{2\Lambda}}{r^2}\left(1 - e^{-2\Lambda}\right) + \frac{2\phi'}{r}$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda}\left[\phi'' + (\phi')^2 + \frac{\phi'}{r} - \phi'\Lambda' - \frac{\Lambda'}{r}\right]$$

$$G_{\phi\phi} = \sin^2\theta\, G_{\theta\theta}$$

For an anisotropic fluid nonzero components of energy-momentum tensor are given by

$$T_{00} = \rho e^{2\phi}$$

$$T_{rr} = p_r e^{2\Lambda}$$

$$T_{\theta\theta} = p_t r^2$$

$$T_{\phi\phi} = \sin^2\theta\, T_{\theta\theta}$$

where $p_r$ and $p_t$ are radial pressure and tangential pressure respectively. Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ provide the following three independent equations,

$$8\pi\rho = \frac{1}{r^2} - e^{-2\Lambda}\left(\frac{1}{r^2} - \frac{2\Lambda'}{r}\right) \quad (2)$$

$$8\pi\, p_r = -\frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) + \frac{2e^{-2\Lambda}\phi'}{r} \quad (3)$$

$$8\pi\, p_t = e^{-2\Lambda}\left(\phi'' + \phi'^2 - \Lambda'\phi' + \frac{\phi' - \Lambda'}{r}\right) \quad (4)$$

From (3) and (4) we obtain the equation

$$8\pi(p_r - p_t) = e^{-2\Lambda}\left(-\phi'' - \phi'^2 + \frac{\phi'}{r} + \phi'\Lambda' + \frac{\Lambda'}{r} + \frac{1 - e^{-2\Lambda}}{r^2}\right) \quad (5)$$

Equation (5) can be expressed as

$$y' - \frac{2\left(w + rw' - r^2 w''\right)}{r(rw' + w)}y = \frac{-2w\left(1 + r^3\Pi(r)\right)}{r(rw' + w)} \quad (6)$$

where

$$y = e^{-2\Lambda},\ w = e^{\phi} \text{ and } \Pi(r) = 8\pi(p_r - p_t)$$

Mass function m(r) is given by

$$m(r) = \frac{r}{2}(1 - e^{-2\Lambda}) \quad (7)$$

In terms of the mass function equation (2) can be written as

$$m'(r) = 4\pi r^2 \rho \quad (8)$$

Equation (6) can be solved for y(r) if w(r) and $\Pi(r)$ are specified. Density $\rho$ and pressures $p_r$ and $p_t$ can be found by putting these in equations (2) – (4).

Equation (6) can be rewritten as

$$u' + \frac{2\left(r^2\phi'' + r^2\phi'^2 - r\phi' - 1\right)}{r(r\phi' + 1)}u = \frac{r^2\Pi(r) + r^2\phi'' + r^2\phi'^2 - r\phi'}{r(r\phi' + 1)} \quad (9)$$

where $u = \frac{m}{r}$. Equation (9) can be regrouped as

$$\phi'' = \frac{ru' - 2u - r^2\Pi(r)}{r^2(1 - 2u)} + \frac{ru' - 2u + 1}{r(1 - 2u)}\phi' - \phi'^2 \quad (10)$$

Equation (10) is a Riccati type differential equation of the form

$$v'(r) = P(r) + Q(r)v - v^2 \quad (11)$$

where

$$v = \phi',\ P(r) = \frac{ru' - 2u - r^2\Pi(r)}{r^2(1 - 2u)},\ Q(r) = \frac{ru' - 2u + 1}{r(1 - 2u)} \quad (12)$$

## 3. Review of Solution Generating Techniques

Einstein equations governing static spherically symmetric perfect fluid distributions provide three independent differential equations for four unknown functions. Solution of such a system of equations depends on the specification of a single generating function. Whereas specification of a generating function provides a solution, there is no guaranty that the resulting solutions will be a realistic one. Lake [6] has given a prescription for obtaining realistic solutions where he has shown that, the necessary condition for the resulting solution to be realistic is that the input function should be a monotonically increasing function of r with a regular minima at r = 0. The formalism has been extended to the case of static anisotropic fluid spheres by Herrera et al [9] which depend on two input functions. To see how it works let us rewrite equation (6) in the following way,

$$y' + \left( \frac{2z'}{z} + 2z - \frac{6}{r} + \frac{4}{r^2 z} \right) y = -\frac{2}{z} \left( \frac{1}{r^2} + \Pi(r) \right) \quad (13)$$

where $w = e^{\int (z(r) - \frac{1}{r}) dr}$

Equation (13) is a first order linear differential equation in y(r) which depends on two functions z(r) and $\Pi(r)$. In closed form solution of (13) can be written as

$$y = \frac{r^6 \left[ -2\int \frac{z(1+\Pi r^2) e^{\int (\frac{4}{r^2 z} + 2z) dr}}{r^8} dr + c \right]}{z^2 e^{\int (\frac{4}{r^2 z} + 2z) dr}} \quad (14)$$

If $\Pi(r) = 0$, solution (14) coincides with that of [6]. The input function $\phi(r)$ in [6] is related to z(r) by $\phi(r) = \int (z - \frac{1}{r}) dr$.

Specifications of two input functions can be reduced to one provide that one additional ansatz is given. For example, if we consider conformally flat anisotropic solutions then conformally flat condition imposes the restriction

$$\phi'' + (\phi')^2 - \phi'\Lambda' + \frac{\phi' - \Lambda'}{r} + \frac{1 - e^{-2\Lambda}}{r^2} = 0 \quad (15)$$

Equation (15) has the solution

$$e^{\phi} = Cr \cosh\left( \int e^{\Lambda(r)} dr \right) \text{ or, } \int (z - \frac{1}{r}) dr = Cr \cosh\left( \int e^{\Lambda(r)} dr \right) \quad (16)$$

From (16) we obtain

$$z(r) = \frac{2}{r} + \frac{e^{\Lambda}}{r} \tanh\left( \int \frac{e^{\Lambda}}{r} dr \right) \quad (17)$$

Putting (15) in (5) we obtain

$$\Pi(r) = r \left( \frac{2e^{-2\Lambda}\Lambda'}{r^2} + 2\frac{e^{-2\Lambda} - 1}{r^3} \right) = r \left( \frac{1 - e^{-2\Lambda}}{r^2} \right)' \quad (18)$$

Any specification of z(r) determines $\Lambda(r)$ through equation (17). Then $\Pi(r)$ is determined by (18). Thus the system is completely determined a single generating function is specified.

Following the prescription given by Herrera et al [9] any static spherically symmetric anisotropic solution can be generated by specifying two input functions z(r) and $\Pi(r)$. Whereas any specification of z(r) and $\Pi(r)$ provides a solution there is no guaranty that the resulting solution will be a realistic one. For a solution to be realistic the weak energy condition $\rho \geq 0$, $\rho + p_t \geq 0$, $\rho + p_r \geq 0$ and the strong energy condition $\rho + p_t \geq 0$, $\rho + p_r \geq 0$, $\rho + 2p_t + p_r \geq 0$ are required to be satisfied. Also regularity at r = 0 requires $p_r(0) = p_t(0) = p_r'(0) = 0$. Lake [10] has presented a less general technique for obtaining realistic anisotropic solutions which requires the specification of only one input function, the density $\rho(r)$ itself. The technique uses the Newtonian equation of hydrostatic equilibrium for an isotropic fluid sphere. To see how it works let us suppose that $\rho$ and $p_r$ are related as follows,

$$p_r' = -\frac{m\rho}{r^2} \quad (19)$$

From (3) we obtain

$$\phi' = \frac{m + 4\pi r^3 p_r}{r(r - 2m)} \quad (20)$$

where $m(r) = \frac{r}{2}(1 - e^{-2\Lambda})$. The generalized Oppenheimer-Volkov equation can be written as

$$p_t = \frac{r}{2}\{p_r' + (\rho + p_r)\phi'\} + p_r \quad (21)$$

Inserting (19) and (20) into (21) we obtain

$$p_t = -\frac{m\rho}{2r} + \frac{(\rho + p_r)(m + 4\pi r^3 p_r)}{2(r - 2m)} + p_r \quad (22)$$

Let us recall that equation (2) can be written as

$$m'(r) = 4\pi r^2 \rho \quad (23)$$

Hence any specification of $\rho(r)$ determines m(r) [hence $\Lambda(r)$], through equation (23) and $p_r$ through equation (19).

Knowing m(r) and $p_r$ we get $\phi(r)$ from equation (20). Thus the system is completely determined if density $\rho$ is specified. If we choose $\rho > 0$ then from equations (19) and (22) it can be seen that

$$p_t \geq p_r \geq 0$$

As a result weak and strong energy conditions are automatically satisfied as long as $\rho > 0$. The formalism is demonstrated below with the help of two examples.

(i) Let $\rho$ = b = constant > 0. From (23) and (19) we respectively get

$$m(r) = \frac{4\pi b r^3}{3} \tag{24}$$

$$p_r{}' = -\frac{4\pi b^2 r}{3} \tag{25}$$

From (25) we obtain

$$p_r = c - \frac{2\pi b^2 r^2}{3} \tag{26}$$

Let r = R be the surface of the fluid sphere so that $p_r(R) = 0$ $\Rightarrow c = \dfrac{2\pi b^2 R^2}{3}$. Putting the value of c in (26) we get

$$p_r = \frac{2}{3}\pi b^2 (R^2 - r^2) \tag{27}$$

Metric function $\Lambda(r)$ is given by

$$e^{-2\Lambda} = 1 - \frac{2m}{r} = 1 - \frac{8\pi b r^2}{3}$$

From (20) we obtain $\phi(r)$,

$$\phi(r) = 4\pi b \left\{ (1 + 2\pi b R^2) \int \frac{r\,dr}{3 - 8\pi b r} - 2\pi b \int \frac{r^3\,dr}{3 - 8\pi b r} \right\} \tag{28}$$

Putting (24) and (27) in (22) we obtain

$$p_t = \frac{2\pi b^2}{3(3 - 8\pi b r^2)} \left[ r^2 (4\pi^2 b^2 r^4 - 3 + 8\pi b r^2) - R^2 (4\pi^2 b^2 r^2 R^2 - 3 + 8\pi^2 b^2 r^4) \right] \tag{29}$$

To avoid singularities in $p_t$ we must have

$$R < \sqrt{\frac{3}{8\pi b}} \quad \Rightarrow R < 2M$$

For this solution we get

$$\rho(0) = b\,,\ p_r(0) = p_t(0) = \frac{2\pi^2 b^2 R^2}{3}\,,$$

$$\rho'(0) = p_r'(0) = p_t'(0) = 0\,.$$

(ii) Equation (25) describes a Newtonian polytrope of index zero. A Newtonian polytrope of index n satisfies the equations

$$m'(r) = 4\pi r^2 \rho\,,\ p_r' = -\frac{\rho m}{r^2}$$

with the equation of state

$$p_r = k\rho^{1+\frac{1}{n}} \tag{30}$$

where k is an arbitrary constant. For a polytrope of index n = 1, equation (30) reduces to

$$p_r = k\rho^2 \tag{31}$$

From equation (31) we get

$$\frac{dp_r}{dr} = 2k\rho \frac{d\rho}{dr} \tag{32}$$

From (32) and the second of equations (30) we obtain

$$m(r) = -2kr^2 \frac{d\rho}{dr} \tag{33}$$

From (33) and the first of equations (30) we get

$$\frac{d}{dr}\left( r^2 \frac{d\rho}{dr} \right) + \frac{2\pi}{k} r^2 \rho = 0 \tag{34}$$

Solution of (34) is given by

$$\rho(r) = \alpha \frac{\sin(ar)}{ar}\,,\ (a^2 = \frac{2\pi}{k}\,, \alpha = \text{constant}) \tag{35}$$

To find an anisotropic solution let us specify the density function $\rho(r)$ as in (35) with $\alpha > 0$. Mass function m(r) is then given by

$$m(r) = 4\pi \int_0^r \rho(x) x^2 dx = \frac{4\pi\alpha}{a^3} \{\sin(ar) - ar\cos(ar)\} \tag{36}$$

Metric function $\Lambda(r)$ is given by $\Lambda = -\dfrac{1}{2}\log\left(1 - \dfrac{2m}{r}\right)$.

Inserting (35) into (31) we obtain

$$p_r = k\alpha^2 \frac{\sin^2(ar)}{a^2 r^2} \tag{37}$$

Metric function $\phi(r)$ can be obtained by putting (36) and (37) in (20).

On the surface r = R of the star $p_r(R)=0$. From (37) we obtain

$$\sin(aR)=0 \ \Rightarrow R=\frac{\pi}{a}.$$

Mass M of the fluid sphere is given by

$$M=\frac{4\pi\alpha}{a^3}(aR)=\frac{4}{a}\alpha R^2, \text{(since } a=\frac{\pi}{R}) \tag{38}$$

From (38) we obtain

$$\frac{M}{R}=\frac{4\pi\alpha}{a^2} \tag{39}$$

For the Buchdahl bound to hold we must have $a^2>9\pi\alpha$. Putting (37) and (36) in (22) we obtain

$$p_t=\frac{I(r)}{32\pi r^5 a^8 (r-2m)}$$

where

$$\begin{aligned} I(r) = {} & 256\pi^4 r^4(\rho_0-\rho)^2(\rho_0+\rho)^2+128\pi^3\rho r^3 a^2(a^2r^3(3\rho^2+\rho_0{}^2) \\ & -4r(\rho_0-\rho)(\rho_0+\rho)+2m(\rho_0-\rho)(\rho_0+\rho)) \\ & -32\pi^2 r^2 a^4(6a^2m\rho^2r^3-2\rho_0{}^2r^2-5\rho_0{}^2mr \\ & +13m\rho^2 r-3m^2\rho^2+\rho_0{}^2m^2) \\ & +8\pi\rho mra^6(3a^2mr^3+4r^2+14mr-2m^2) \\ & -a^8m^2(4r^2+10mr-m^2) \end{aligned}$$

where $\rho_0=\rho(0)$.

For this solution $\rho(0)=\alpha$ , $p_r(0)=p_t(0)=\frac{2\pi\alpha}{a^2}$ , $\rho'(0)=p_r'(0)=p_t'(0)=0$,

$\rho(R)=p_r(R)=p_t(R)=p_r'(R)=0$ , $\rho'(R)=-\frac{a\alpha}{\pi}$ along with

$$p_t'(R)=-\frac{16\pi a^3}{a(a^2-8\pi\alpha)}$$

# 4. New Techniques for Generating Solutions

In this section we provide three different techniques for generating static spherically symmetric anisotropic solutions.

*I. Anisotropic Solutions from Known Anisotropic Solutions*

Let us suppose that $(u_0,v_0)$, where $u_0=\frac{m_0(r)}{r}$ and $v_0=\phi'$, is a known anisotropic solution. Then $(u_0,v_0)$ satisfies equation (11). Therefore we have

$$v_0'=P_0(r)+Q_0(r)v_0-v_0{}^2 \tag{40}$$

where $P_0(r)=\frac{ru_0'-2u_0-r^2\Pi_0(r)}{r^2(1-2u_0)}$,

$$Q_0(r)=\frac{ru_0'-2u_0+1}{r(1-2u_0)}$$

Here $r^2\pi_0(r)$ is obtained by using $(u_0,v_0)$ in (3) and (4). From (40) we find that $v(r)=v_0(r)$ is a particular solution of the Riccati differential equation

$$v'=P_0(r)+Q_0(r)v-v^2 \tag{41}$$

Let $V=v_0+\frac{1}{z(r)}$ be the general solution of (41) so that we have

$$V'=P_0(r)+Q_0(r)V-V^2 \tag{42}$$

Equation (42) can be regrouped as

$$u_0'+\frac{2\left(r^2V'+r^2V^2-rV-1\right)}{r(rV+1)}u_0=\frac{r^2\Pi_0(r)+r^2V'+r^2V^2-rV}{r(rV+1)} \tag{43}$$

Therefore $u=u_0$ is a particular solution of the differential equation

$$u'+\frac{2\left(r^2V'+r^2V^2-rV-1\right)}{r(rV+1)}u=\frac{r^2\Pi_0(r)+r^2V'+r^2V^2-rV}{r(rV+1)} \tag{44}$$

Putting $u=u_0+U(r)$ in (44) and using (43) we obtain the equation

$$U'+\frac{2\left(r^2V'+r^2V^2-rV-1\right)}{r(rV+1)}U=0 \tag{45}$$

Therefore if $(u_0,v_0)$ is a known anisotropic solution then $(u_0+U,v_0+\frac{1}{z})$ is also an anisotropic solution where $v_0+\frac{1}{z}$ is the solution of (41) and $U(r)$ is the solution of (45).

*II. Anisotropic Solutions from Isotropic Solutions*

Let us suppose that $(u_0,v_0)$ is a known static spherically symmetric isotropic solution i.e. we let

$$v_0'=P_0(r)+Q_0(r)v_0-v_0{}^2 \tag{46}$$

where

$$P_0(r) = \frac{ru_0' - 2u_0}{r^2(1-2u_0)} \text{ and } Q_0(r) = \frac{ru_0' - 2u_0 + 1}{r(1-2u_0)} \quad (47)$$

Proceeding as in I above we get

$$u_0' + \frac{2\left(r^2V' + r^2V^2 - rV - 1\right)}{r(rV+1)} u_0 = \frac{r^2V' + r^2V^2 - rV}{r(rV+1)} \quad (48)$$

where $V = v_0 + \frac{1}{z(r)}$ is the general solution of

$$v' = P_0(r) + Q_0(r)v - v^2 \quad (49)$$

where $P_0(r)$ and $Q_0(r)$ are given by (47). Therefore $u = u_0$ is a particular solution of the differential equation

$$u' + \frac{2\left(r^2V' + r^2V^2 - rV - 1\right)}{r(rV+1)} u = \frac{r^2V' + r^2V^2 - rV}{r(rV+1)} \quad (50)$$

Let us consider the equation

$$u' + \frac{2\left(r^2V' + r^2V^2 - rV - 1\right)}{r(rV+1)} u = \frac{r^2\Pi(r) + r^2V' + r^2V^2 - rV}{r(rV+1)} \quad (51)$$

where $\Pi(r)$ is arbitrary. Equation (51) reduces to (50) if $\Pi(r) = 0$ . Putting $u = u_0 + U(r)$ in (51) and using (48) we get the equation

$$U' + \frac{2\left(r^2V' + r^2V^2 - rV - 1\right)}{r(rV+1)} U = \frac{r^2\Pi(r)}{r(rV+1)} \quad (52)$$

Therefore if $(u_0, v_0)$ is a given isotropic solution, then $(u_0 + U, v_0 + \frac{1}{z})$ is an anisotropic solution where $v_0 + \frac{1}{z}$ is the solution of (49) and U(r) is the solution of (52). Any specification of $\Pi(r)$ generates a static spherically symmetric anisotropic solution. However not all such solutions will be realistic. For the resulting solution to be realistic $r^2\Pi(r)$ must satisfy some conditions.

*III. Anisotropic Solutions Satisfying an Ansatz*

For the equation (6), we consider the class of solutions which satisfy the ansatz

$$\frac{w + rw' - r^2w''}{r(rw' + w)} = \frac{w(1 + r^3\Pi(r))}{r(rw' + w)} \quad (53)$$

Equation (53) can be rearranged as

$$rw'' - w' + J(r)w = 0 \quad (54)$$

where $J(r) = r^2\Pi(r)$. Equation (54) is a second order linear differential equation in w(r) and can be solved if J(r) is specified. Therefore all such solutions depend on the specification of a single generating function J(r). Whereas any specification of J(r) generates a solution not all such solutions will be realistic. To obtain realistic solutions J(r) should be chosen intelligently. Regularity at the origin r = 0 requires J(0) = 0.

## 5. New Solution

We have found a new solution using technique-III described in Section-4. We recall that any specification of J(r) determines w = exp ($\phi$) through equation (6) and thereby generates a solution. Regularity at r = 0 requires J(0) = 0. Keeping this in mind we choose

$$J(r) = \frac{a^2r^3}{(ar^2+1)^2} \quad (55)$$

Inserting (55) into (54) we obtain the differential equation

$$r(ar^2+1)^2w'' - (ar^2+1)^2w' + a^2r^3w = 0 \quad (56)$$

Under transformation $x = \log(ar^2+1)$ equation (56) reduces to

$$\frac{d^2w}{dx^2} - \frac{dw}{dx} + \frac{w}{4} = 0 \quad (57)$$

Equation (57) has the solution

$$w = e^{\phi} = (c_1 + c_2x)e^{\frac{x}{2}} = (c_1 + c_2\log(ar^2+1))\sqrt{ar^2+1} \quad (58)$$

Solution of equation (6) is given by

$$y = e^{-2\Lambda} = 1 + Ce^{-\int f(r)dr} \quad (59)$$

where

$$f(r) = \frac{-2w(1+rJ)}{r(rw'+w)} \quad (60)$$

Putting (55) and (58) in (60) we obtain

$$f(r) = \frac{-2\left\{c_1 + c_2\log(ar^2+1)\right\}\left(2a^2r^4 + 2ar^2 + 1\right)}{r(ar^2+1)\left\{2(c_1+c_2)ar^2 + c_2(2ar^2+1)\log(ar^2+1) + c_1\right\}} \quad (61)$$

Evaluation of $\int f(r)dr$ is not easy. So we consider the particular solution of (57) obtained by letting $c_2 = 0$. Then we have

$$f(r) = \frac{-2\left(2a^2r^4 + 2ar^2 + 1\right)}{r(ar^2+1)(2ar^2+1)} \tag{62}$$

Integration of (62) is found to be given by

$$\int f(r)dr = \log \frac{2ar^2+1}{r^2(ar^2+1)} \tag{63}$$

Putting (63) in (59) we obtain

$$e^{-2\Lambda} = 1 + \frac{Cr^2(ar^2+1)}{2ar^2+1} \tag{64}$$

Therefore we get the solution

$$ds^2 = -c_1^2(ar^2+1)dt^2 + \frac{dr^2}{1+\frac{Cr^2(ar^2+1)}{2ar^2+1}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{65}$$

## 6. Properties of the Solution

For this solution density and pressure are given by

$$8\pi\rho = \frac{3k}{2} + \frac{1}{3}\left\{\frac{1}{2ar^2+1} + \frac{2}{(2ar^2+1)^2}\right\}$$

$$8\pi p_r = \frac{2a}{ar^2+1} - \frac{k\left(3ar^2+1\right)}{2ar^2+1}$$

$$8\pi p_t = \frac{2a}{ar^2+1} - \frac{k\left(3ar^2+1\right)}{2ar^2+1} - \frac{a^2r}{(ar^2+1)^2}$$

where k = - c. Central density and central pressures are given by

$$8\pi\rho(0) = \frac{3k}{2} + 1 > 0$$

$8\pi p_r(0) = 8\pi p_t(0) = 2a - k > 0$, if k > 0 and a > $\frac{k}{2}$

$\frac{d\rho}{dr}$ and $\frac{dp}{dr}$ are given by

$$\frac{d\rho}{dr} = -\frac{ar(2ar^2+5)}{6\pi(2ar^2+1)^3} < 0$$

$$\frac{dp_r}{dr} = -\frac{ar}{2\pi(ar^2+1)^2}\left\{a + \frac{k(ar^2+1)^2}{2a(2ar^2+1)^2}\right\} < 0$$

Therefore both $\rho(r)$ and $p_r(r)$ are decreasing functions of r. $\rho'(0) = p_r(0) = 0$.

Let r = R be the surface of the fluid sphere. Then $p_r(R) = 0$. This gives

$$R = \left[\frac{4a^2 - 3ak - 1 + \sqrt{(4a^2-3ak-1)^2 + 12a^2k(2a-1)}}{6a^2k}\right]^{\frac{1}{2}}$$

If we choose $k = \frac{4a^2-1}{3a}$ then we get $R = \{a(2a+1)\}^{-\frac{1}{4}}$

Let M be the mass of the fluid sphere. Since the interior solution must be joined smoothly onto the vacuum Schwarzschild solution at r = R we must

$$e^{-\lambda(R)} = 1 - \frac{2M}{R}$$

This implies $\frac{M}{R} = \frac{kR(aR^2+1)^2}{2a(2aR^2+1)^2}$

$$= \frac{\sqrt{4a^2-1}\left(\sqrt{(2a-1)a} + \sqrt{4a^2-1}\right)}{6a\left(2a+\sqrt{a(2a+1)}\right)}$$

Buchdahl condition $\frac{M}{R} < \frac{4}{9}$ is satisfied if

$$\frac{\sqrt{4a^2-1}\left(\sqrt{(2a-1)a} + \sqrt{4a^2-1}\right)}{2a\left(2a+\sqrt{a(2a+1)}\right)} < \frac{4}{3}$$

$$\Leftrightarrow 8a\left\{2a+\sqrt{a(2a+1)}\right\} > 3(2a-1)\sqrt{a(2a+1)} + 12a^2 - 3$$

$$\Leftrightarrow 4a^2 + (2a+3)\sqrt{a(2a+1)} + 3 > 0$$

which is satisfied for all a > 0.

Therefore Buchdahl condition is satisfied if a > 0.

## 7. Conclusions

Generation of static spherically symmetric anisotropic solution needs the specification of two input functions. Proper choice of two input functions for generating a realistic solution is not easy. We have described three techniques for generating static spherically symmetric anisotropic solutions which need the specification of one input function instead of two. Using technique III we have found a physically acceptable solution. In a future work we hope to find the conditions satisfied by an input function so that the resulting solution may be realistic. The content of this work would make a useful addition to a physical solutions of Einstein's equations.

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