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A Reduced Differential Transform Method for Solving the Advection and the Heat-Like Equations

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Abstract

In this study, we apply Reduced Differential Transform Method (RDTM) for solving partial differential equations in different dimensions with variable coefficients. RDTM is employed to obtain the exact solution of simple homogeneous advection and heat-like equations. The RDTM produces a solution with few and easy computation. The method is simple, accurate and efficient.

Keywords

Reduced Differential Transform Method, Advection Equation, Heat-Like Equations, Analytic Solutions

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1. Introduction

The importance of research on partial differential-algebraic equations (PDAEs) is that many phenomena, practical or theoretical, can be easily modelled by such equations. Linear and nonlinear PDAEs are characterized by means of indices which play an important role in the treatment of these equations. The differentiation index is defined as the minimum number of times that all or part of the PDAE must be differentiated with respect to time, in order to obtain the time derivative of the solution, as a continuous function of the solution and its space derivatives (Ben Benhammouda et *al*., 2014). Many physical problems are described by mathematical models with partial differential equations (Kanzari et *al*. 2012). A mathematical model is a simplified description of physical reality expressed in mathematical terms (Kanzari et *al*., 2015; Ben Mariem and Ben Mabrouk, 2014). Thus, the investigation of the exact or approximation solution helps us to understand the means of these mathematical models (Kanzari and Ben Mariem, 2014; Ben Mariem and Ben Mabrouk, 2014). Several numerical methods were developed for solving partial differential equations with variable coefficients such us He's Polynomials (Mohyud-Din, 2009), the homotopy perturbation method (Jin 2008), homotopy analysis method (Alomari et al., 2008) and the modified variational iteration method (Noor and Mohyud-Din, 2008). In this paper RDTM is used to obtain the exact solution of simple homogeneous advection and the heat-like equations in the forms:

2. RDTM Method

The basic definitions of reduced differential transform method are introduced as follows: If the function $f(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the studied domain, then let:

$$F_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} f(x,t)\right]_{t=0} \tag{1}$$

where the t-dimensional spectrum function $F_k(x)$ is the

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transformed function.

The differential inverse transform of $F_k(x)$ is defined as:

$$f(x,t)=\sum_{k=0}^{\infty}F_k(x)t^k \tag{2}$$

If we combine (1) and (2) we can write:

$$f(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}f(x,t)\right]_{t=0}t^k \tag{3}$$

From the above definitions, the concept of the reduced differential transform is derived from the power series expansion. The fundamental mathematical operations performed by RDTM (Keskin and Oturanc, 2010; Sohail and Mohyud-Din, 2012 (a); Sohail and Mohyud-Din, 2012 (b)) can be readily obtained and are listed in Table 1.

Table 1. Reduced differential transform.

Functional Form	Transformed Form
$f(x,t)$	$F_k(x)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}f(x,t)\right]_{t=0}t^k$
$h(x,t)=f(x,t)\pm g(x,t)$	$H_k(x)=F_k(x)\pm G_k(x)$
$h(x,t)=\alpha.f(x,t)$	$H_k(x)=\alpha.F(x,t)$; α is a constant
$h(x,t)=x^m t^n$	$H_k(x)=x^m\delta(k-n),\ \delta(k)=\begin{cases}1, & k=0\\ 0, & k\neq 0\end{cases}$
$h(x,t)=x^m t^n f(x,t)$	$H_k(x)=x^m F_{k-n}(x)$
$h(x,t)=f(x,t).g(x,t)$	$H_k(x)=\sum_{r=0}^{k}G_r F_{k-r}(x)=\sum_{r=0}^{k}F_r G_{k-r}(x)$
$h(x,t)=\frac{\partial^r}{\partial t^r}f(x,t)$	$H_k(x)=(k+1)...(k+r)F_{k+r}(x)$
$h(x,t)=\frac{\partial}{\partial x}f(x,t)$	$H_k(x)=\frac{\partial}{\partial x^2}F_k(x)$
$h(x,t)=\frac{\partial^2}{\partial x^2}f(x,t)$	$H_k(x)=\frac{\partial^2}{\partial x^2}F_k(x)$

3. Numerical Applications

3.1. Advection Equation

Consider the homogeneous advection equation given by (Alomari et *al.*, 2008) as:

$$f_t+ff_x=0,\quad f(x,0)=-x \tag{4}$$

$f_t=-ff_x$, now taking the reduced differential transform of (2):

$$(k+1)F_{k+1}=-\sum_{r=0}^{k}F_r\frac{\partial}{\partial x}F_{k-r} \tag{5}$$

with $F_0(x)=-x$ we can then obtain $F_k(x)$ values successively as $F_1(x)=F_2(x)=F_3(x)=...=F_k(x)=-x.$ Using the differential inverse transform (2) we have:

$$f(x,t)=-x\sum_{n=0}^{\infty}t^n \tag{6}$$

equation (6) is a Taylor series that converges to:

$$f(x,t)=\frac{x}{t-1} \tag{7}$$

under $|t|<1$ which is the exact solution (Figure 1).

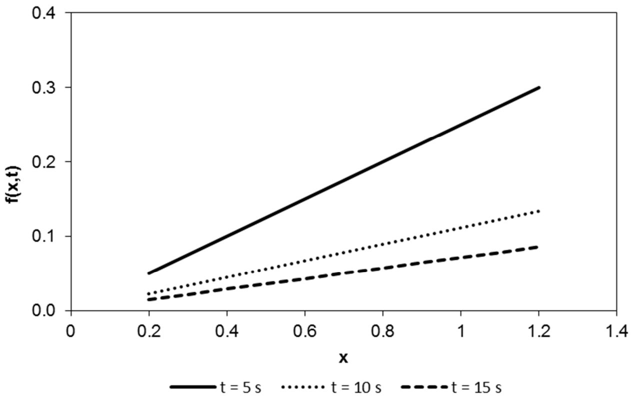

Figure 1. Example of the advection equation in 1D with three different step times and for x = 1.2.

3.2. Heat-Like Equations

Consider the one-dimensional initial value problem which describes the heat-like equations:

$$f_t(x,t)=\frac{x^2}{2}f_{xx}(x,t),\quad f(x,0)=x? \tag{8}$$

Taking the reduced differential transform of (8):

$$(k+1)F_{k+1}(x)=\frac{x^2}{2}\frac{\partial^2}{\partial x?}F_k(x) \tag{9}$$

where the t-dimensional spectrum function $F_k(x)$ is the transform function. From the initial condition, we can write:

$$F_0(x)=x^2 \tag{10}$$

Substituting (10) into (9) we obtain the following $F_k(x)$ values successively:

$$F_1(x)=x?$$
$$F_2(x)=\frac{1}{2}x?$$
$$F_3(x)=\frac{1}{6}x?$$
$$F_4(x)=\frac{1}{24}x?$$
$$...$$

Finally the differential inverse transform (8) of $F_k(x)$ gives:

$$f(x,t)=\sum_{k=0}^{\infty}F_k(x)t^k=x^2\sum_{k=0}^{\infty}\frac{t^k}{k!}=x^2e^t \tag{11}$$

which is the exact solution (Figure 2).

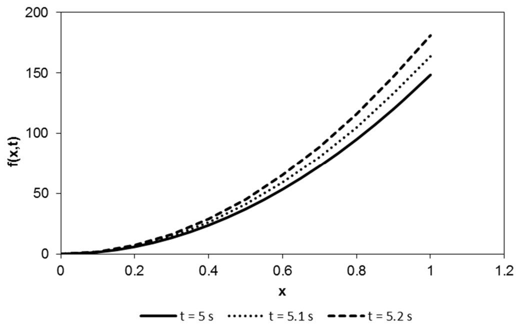

Figure 2. Example of the heat-like equation in 1D with three different step times and for x = 1.2.

Consider the two-dimensional initial value problem which describes the heat-like equations:

$$f_t(x,t)=\frac{y^2}{2}f_{xx}(x,y,t)+\frac{x^2}{2}f_{yy}(x,y,t),\quad f(x,y,0)=y? \tag{12}$$

Applying the reduced differential transform of (12):

$$(k+1)F_{k+1}(x,y)=\frac{y^2}{2}\frac{\partial^2}{\partial x?}F_k(x,y)+\frac{x^2}{}\frac{\partial^2}{\partial y}F_k(x,y) \tag{13}$$

where the t-dimensional spectrum function $F_k(x,y)$ is the transform function. From the initial condition, we can write:

$$F_0(x,y)=y? \tag{14}$$

Substituting (14) into (13) we obtain the following $F_k(x,y)$ values successively:

$$F_1(x,y)=x?$$
$$F_2(x,y)=\frac{y^2}{2}$$
$$F_3(x,y)=\frac{x^2}{6}$$
$$F_4(x,y)=\frac{y^2}{24}$$
$$F_5(x,y)=\frac{x^2}{120}$$
$$\ldots$$

By using definition (3), we obtained the closed from series solution as

$$f(x,y,t)=\sum_{k=0}^{\infty}F_k(x,y)t^k=y^2\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots\right)+x^2\left(1+\frac{t^3}{3!}+\frac{t^5}{5!}+\ldots\right)=y^2\cosh(t)+x^2\sinh(t) \tag{15}$$

which is the exact solution (Figure 3).

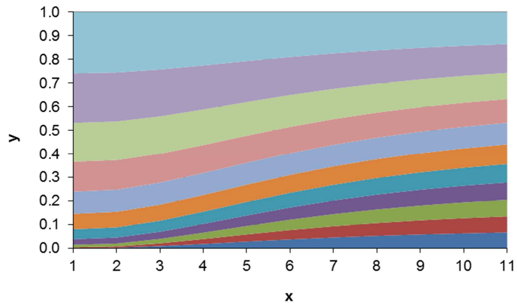

Figure 3. Example of the heat-like equation in 2D with a time step of 5s.

Consider the three-dimensional initial value problem which describes the heat-like equations:

$$f_t(x,t)=(x,y,z)^4+\frac{1}{36}\left(x?f_{xx}+y\ f_{yy}+z\ f_{zz}\right) \tag{16}$$

with initial condition:

$$f(x,y,z,0)=0 \tag{17}$$

Similarly, by using the RDTM, we obtain the recurrence equation:

$$(k+1)F_{k+1}(x,y,z)=(xyz)^4\,\delta(k)+\frac{1}{36}\left[x^2\frac{\partial^2}{\partial x^2}F_k(x,y,z)+y^2\frac{\partial^2}{\partial y^2}F_k(x,y,z)+z^2\frac{\partial^2}{\partial z^2}F_k(x,y,z)\right] \tag{18}$$

From the initial condition:

$$F_0(x,y,z)=0 \tag{19}$$

We have, by substitution:

$$F_1(x,y,z)=(x,y,z)^4$$
$$F_2(x,y,z)=\frac{1}{2}(x,y,z)^4$$
$$F_3(x,y)=\frac{1}{6}(x,y,z)^4$$
$$F_4(x,y)=\frac{1}{24}(x,y,z)^4$$
$$\ldots$$

The series solution is given by:

$$f(x,y,z,t)=(xyz)^4\left[t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\right]=(xyz)^4(e^t-1) \tag{20}$$

which is the exact solution.

4. Discussion

In this paper, we presented the reduced differential transform method (RDTM) as a useful analytical tool to solve partial differential-algebraic equations (PDAEs). The RDTM solution procedure does not involve unnecessary computation like that related to noise terms, which is a common problem for approximation methods like the HPM or others. This property of RDTM greatly reduces the volume of computation and improves the efficiency of the method. On the one hand, semianalytic methods like HPM, HAM, and VIM, among others, require an initial approximation for the solutions sought and the computation of one or several adjustment parameters (Benhammouda et *al*., 2014). If the initial approximation is properly chosen, then the results can be highly accurate. Nonetheless, there is no general method to choose such initial approximation. This issue motivates the use of adjustment parameters obtained by minimizing the leastsquares error with respect to the numerical solution. On the other hand, RDTM method does not require any trial equation or a procedure for least-squares error minimization. As well, RDTM obtains its coefficients using an easily computable straightforward procedure that can be implemented into programmes. In the end it is very important to mention that the treatment of higher-index PDAEs is still an open issue in science and requires further research.

5. Conclusion

In this study, reduced differential transform method has been applied to solving the advection and heat-like equations. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. The result show RDTM needs small size of computation contrary to other numerical methods (classical differential transform method (DTM), Adomain method and homotopy perturbation method. Hence, this method is a powerful and an efficient technique in finding the exact solutions for wide classes of problems, also the speed of the convergence is very fast.

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