Physics Journal
Vol. 2, No. 1, 2016, pp. 1-5
http://www.aiscience.org/journal/pj

Single Measurement Bell's Inequality with Detection Efficiencies

Vladimir K. Ignatovich*

Frank Laboratory of Neutron Physics of the Joint Institute for Nuclear Research, Dubna, Russia

Abstract

Within the EPR paradox for polarizations of particles the question is formulated: whether a source of pairs of oppositely flying particles emits them in an entangled state or as individual particles with their own polarizations. To answer this question the simplest Bell's inequality for photons is proposed. It can be checked in a single measurement. The outcome of the possible measurement with account of experimental limitations is calculated.

Keywords

Quantum Mechanics, Uncertainty Relations, EPR Paradox, Bell's Inequality, Hidden Variables, Photon Polarization

Received: September 8, 2015 / Accepted: November 8, 2015 / Published online: December 6, 2015

1. Introduction

In the EPR paradox, formulated for polarizations of particles [1] the main question is: if a source emits pairs of particles flying in opposite directions, are these particles emitted in an entangled state, or they are independent individual particles with their own polarizations. To answer this question the Bell's inequality was devised [2-4]. It suggests four coincident measurements by two analysers of polarization with four different orientations of their axes. The performed experiments [5-7] even declared to show evidence in favour of entanglement, though there is some scepticism [8] that these results are loop hole free. It was reported recently [9] that Bell's inequality can be reduced to the simplest form, which can be checked in a single measurement. This inequality was derived for a model of a source emitting two photons with parallel polarizations flying in opposite directions to experimenters Alice and Bob. The inequality helps to check, whether emitted photons are individual ones with their premeasurable polarizations, or they are in some entangled state, and their polarization does appear only after measurement by one of the experimenters. However the inequality in [9] was derived for an ideal experiment with unit efficiency of detectors and analysers. In real experiments there are a lot of limitations, which must be taken into account. It is done in this paper. Below the EPR paradox and derivation of the ideal simplest Bell's inequality are reminded, and then the Bell's inequality corrected with account of different limitations of a real experiment is presented.

2. EPR Paradox with Photon Pairs

First, let us remind the Bohm-Aharonov version [1] of the EPR paradox with photons. Consider a photon pair, which is emitted by one of atoms located at the point $z = 0$. Photons travel in opposite directions along z-axis. One photon goes to the point $z = A$ to the experimenter Alice, and the other one goes to the point $z = -B$ to the experimenter named Bob, as shown in Fig. 1. Photons have the same direction of the linear polarization, but this direction is uncertain. The unit vector along them have a uniform distribution over the azimuthal angle in the plane (x, y), perpendicular to the direction of the photons propagation. Alice and Bob are using birefringent crystals of calcite to measure the polarization of

* Corresponding author
E-mail address: v.ignatovi@gmail.com (V. K. Ignatovich)

the photons arriving to them. A calcite crystal has two mutually perpendicular axes: ordinary one (denoted by direction of the unit vector **o**) and extraordinary one (denoted by **n**). Assume that both axes lie in the plane (*x, y*). If a coming to a calcite crystal photon is polarized along **o** axis, it is registered by detector D_1, and if the coming photon is polarized along **n**, then it is registered by the detector D_2. Axes of the crystals can be rotated around the z-axis.

The EPR paradox can be illustrated as follows. Imagine that the two calcites of Alice and Bob are oriented parallel to each other, and Alice is somewhat closer to the source of photons than Bob, i.e. she is the first who measures photon of the pair. Imagine two photons flying to A and B are polarized along a vector $\boldsymbol{\gamma}$, which is not parallel to $\mathbf{o}_A$ or $\mathbf{n}_A$ of Alice's calcite, i.e. $\boldsymbol{\gamma} = \alpha\mathbf{o}_A + \beta\mathbf{n}_A$, where α and β are the coordinates of the vector $\boldsymbol{\gamma}$ in the basis of two orthogonal unit vectors. With probability $|\alpha|^2$ the photon at Alice is registered by the detector D_{1A}, and with probability $|\beta|^2$ by the detector D_{2A}, but it is counted only by a single detector. Let's say it is registered by the detector D_{1A}. Then, according to the nonlocal quantum mechanics, since the two photon must have the same polarization, the photon flying to Bob instantly becomes polarized along $\mathbf{o}_A$, and, as Bob's calcite is oriented parallel to that of Alice, the photon will be with certainty registered only by D_{1B}, though without Alice's measurement the photon at Bob could with probability $|\beta|^2$ be registered by the detector D_{2B}. It means that measurement of Alice instantly changes photon polarization at Bob, no matter how great is the distance $A + B$ there between. This mental phenomenon, sometimes called nonlocality, is the essence of the EPR paradox. From this it follows that either the theory of relativity is not true, because there are signals propagating with arbitrarily high speed, or quantum mechanics is not complete, i.e. it is necessary to provide to it additional yet unknown, i.e. hidden parameters.

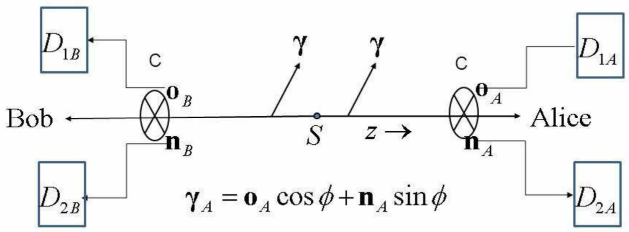

Figure 1. An experiment to demonstrate the EPR paradox in nonlocal quantum mechanics. S is the source of photon pairs with parallel polarizations γ, flying in two opposite directions to the experimenters Alice and Bob. C - birefringent calcite crystals, analysing polarization. The crystals have an ordinary, **o**, and extraordinary, **n**, axes. Photons polarized along **o**, are registered by detectors D_1, and photons polarized along **n**, are recorded by detectors D_2. Alice is closer to the source and registers her photon polarization γ_A the first. φ is the azimuthal angle of the polarization vector.

According to quantum mechanics the process described above is not correct, because photons emitted by the source are not individual particles with their pre-existent polarizations. In quantum mechanics the source emits not photons. It emits only a two photon's entangled state described by the wave function

$$|\psi(1,2)\rangle = |\mathbf{x}\rangle_1 |\mathbf{x}\rangle_2 + |\mathbf{y}\rangle_1 |\mathbf{y}\rangle_2 , \quad (1)$$

where **x**, **y** are two arbitrary orthogonal unit vector in the (*x, y*) plane. Such a wave function means that the photons with the same probability amplitude can be polarized along any of two mutually perpendicular vectors **x** and **y**. Only after measurement by Alice the photon flying to Bob acquires its individual polarization. Suppose that Alice's photon is registered, for example, by the detector D_{1A}, then the wave function of the particle moving to Bob, becomes

$$|\psi(2)\rangle = \langle\mathbf{o}_A||\psi(1,2)\rangle = \langle\mathbf{o}_A||\mathbf{x}\rangle_1|\mathbf{x}\rangle_2 + \langle\mathbf{o}_A||\mathbf{y}\rangle_1|\mathbf{y}\rangle_2 = \alpha_A|\mathbf{x}\rangle_2 + \beta_A|\mathbf{y}\rangle_2 , \quad (2)$$

where

$$\alpha_A = \langle\mathbf{o}_A||\mathbf{x}\rangle_1 , \beta_A = \langle\mathbf{o}_A||\mathbf{y}\rangle_1 , \quad (3)$$

i.e. it takes the form of a photon with the definite polarization along the unit vector $\boldsymbol{\gamma}_B = \alpha_A\mathbf{x} + \beta_A\mathbf{y}$, which is parallel to the orientation of the axis $\mathbf{o}_A$ of the Alice's calcite. So, the measurement by Alice instantly identifies Bob's photon polarization, regardless of how far it is, i.e. effect of measurement propagates with arbitrarily high speed.

To test the validity of the above, Alice and Bob can make measurements with a lot of the emitted photon pairs and then compare whether each pair of photons is detected with the same detectors. Of course, it is necessary to mark each emitted photon pair, for instance, by fixing the registration time of photons. However, experiments on the EPR paradox are not carried out in such a way, but by measurement of some inequality to verify as to whether the measurement result contradicts the prediction by Bell [2-3].

3. The Usual Derivation of the Bell's Inequality

We assign the number +1 to a photon, if after an analyser it is registered by detector D_1, and-1, if after the analyser it is registered by detector D_2. Usually Alice and Bob are recommended to conduct 4 experiments: Alice oriented the ordinary axis of her calcite in the direction of two unit vectors **a** and **a'**, and Bob for each of these options of Alice oriented ordinary axis of his calcite along directions **b** and **b'**. Bell's inequality appears as follows [4]: to photons incident on the crystal with the axis **a** it is assigned the stochastic value $a = \pm 1$, and similarly for other axes. Then, denoting

$$M = a(b+b') + a'(b-b')\,, \qquad (4)$$

where all the letters on the right correspond to one of the values ±1, we can write the following obvious equality [10,11]

$$|M| = 2\,. \qquad (5)$$

After averaging of the expression (5) over many photons one gets the inequality

$$-2 < \langle ab\rangle + \langle ab'\rangle + \langle a'b\rangle - \langle a'b'\rangle < 2\,, \qquad (6)$$

where the sign <> denotes averaging. D_{1B}

According to nonlocal quantum mechanics, in the case, when the angle between the axes of **a** and **b** is equal to θab, then, if a photon at Alice side is detected, for example, by the detector D_{1A}, then at Bob side his photon with probability $\cos^2\theta ab$ is counted by the detector D_{1B} and with probability $\sin^2\theta ab$ by the detector D_{2B}, i.e. (for registration by any detector at Alice side) Bob measures the correlation function

$$E(a,b) \equiv \langle ab\rangle = \cos^2\theta_{ab} - \sin^2\theta_{ab} = \cos 2\theta_{ab}\,. \qquad (7)$$

Selecting the azimuthal angles of crystals as is shown in Fig. 2: $\theta\, a = \pi/8$, $\theta\, b = 0$, $\theta\, a' = -\pi/8$, $\theta\, b' = \pi/4$, one obtains

$$\langle ab\rangle + \langle ab'\rangle + \langle a'b\rangle - \langle a'b'\rangle = \frac{4}{\sqrt{2}} > 2\,, \qquad (8)$$

which contradicts the inequality (6).

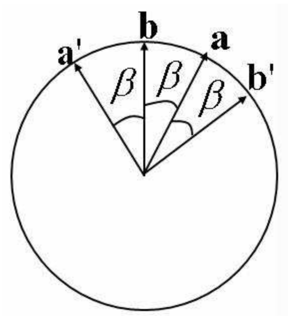

Figure 2. Directions of ordinary axes in Alice and Bob 4 experiments.

However, the inequality (6) is too complicated. It is possible to simplify it so that it can be checked in a single experiment.

4. The Simplest Bell's Inequality

The simplest Bell's inequality for an ideal experiment looks

$$|\langle ab\rangle| < \frac{1}{2}\,. \qquad (9)$$

It immediately follows from (9) that (6) is also valid, but (9) is sufficient.

To prove the inequality (9) imagine that the source emits, as it was tacitly expected at the beginning of this article, the pairs of real photons instead of combination (1). In this case, the wave function of photons can be represented by the product of the two-photon fields

$$|\psi(1,2)\rangle = |\boldsymbol{\gamma}\rangle_1 |\boldsymbol{\gamma}\rangle_2\,. \qquad (10)$$

Their polarization is the same, but has a uniform probability distribution in the azimuthal angle φ ("hidden" parameter) in the plane (x, y). For a given φ the geometry is shown in Fig. 3.

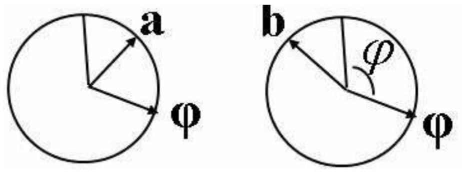

Figure 3. The geometry of orientation of two ordinary axes of Alice's and Bob's calcite and of the photon pair polarization.

In this case, the photon electric field $\boldsymbol{\gamma}_1(\varphi)$, registered by Alice, according to the local quantum mechanics, is characterized by the correlation function

$$P(\mathbf{a},\boldsymbol{\gamma}) = \cos^2(\phi-\theta_a) - \sin^2(\phi-\theta_a) = \cos 2(\phi-\theta_a)\,, \qquad (11)$$

and photons detected by Bob, according to the local quantum mechanics, are characterized by the correlation function

$$P(\mathbf{b},\boldsymbol{\gamma}) = \cos^2(\phi-\theta_b) - \sin^2(\phi-\theta_b) = \cos 2(\phi-\theta_b)\,. \qquad (12)$$

The correlation function of both photons counted by Alice and Bob is

$$E(\mathbf{a},\mathbf{b}) \equiv \langle ab\rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} \cos 2(\phi-\theta_a)\cos 2(\phi-\theta_b) = \frac{1}{2}\cos 2\theta_{ab}\,, \qquad (13)$$

because normalization integral is equal to unity.

The resulting expression differs from (7) only by the constant factor. If it can be extracted, then the violation of the Bell's inequality can be proven in a single measurement at $\theta ab < \pi/6$. If extraction of this factor in experiments is impossible, it is nevertheless possible to distinguish the local quantum mechanics with individual photons (10) and the "hidden" parameter φ from nonlocal quantum mechanics without this parameter and with the wave function (1). In the case of independent photons and parallel aligned crystals counting of a photon by the detector D_{1A} at Alice side does not prohibit registration of the photon at Bob side by the detector D_{2B}. The probability of such registration is

$$P(a_1,b_2) = \int_0^{2\pi} \frac{d\phi}{2\pi} \cos^2(\phi-\theta_a)\sin^2(\phi-\theta_b) = \frac{1}{4}\left(1 - \frac{1}{2}\cos(2\theta_{ab})\right) = \frac{1}{8}\,,$$

while the probability of registration by the same detectors is equal to [7]

$$P(a_1,b_1)=\int_0^{2\pi}\frac{d\phi}{2\pi}\cos^2(\phi-\theta_a)\cos^2(\phi-\theta_b)=\frac{1}{4}\left(1+\frac{1}{2}\cos 2\theta_{ab}\right)=\frac{3}{8}.$$

So for independent photons the ratio of the probability of registration by different detectors to registration by the same detectors at θab =0 is 1/3.

Expression (13) is derived for an ideal experiment, in which every photon pair is registered with unit probability at both sides. Let's now introduce limitations, which are inherent in every experiment.

5. Inequality with Experimental Limitations for Known Rate of Pair Emission by the Source

Let's suppose that the rate of pair emission by the source is well known. Denote the efficiency of detectors at Alice side by $\eta_{a1,2}$, probability of wrong transmission or depolarization on calcite crystal from detector j to detector i by ζ_{aij}, and respectively the similar values at Bob's side. For the individual photons instead of (11) -- (13) one has correlation probability for the Alice's photon to be

$$\begin{aligned}P(\mathbf{a},\boldsymbol{\gamma})&=\eta_{a1}(1-\zeta_{a12})\cos^2(\phi-\theta_a)\\&-\eta_{a2}(1-\zeta_{a21})\sin^2(\phi-\theta_a)+\eta_{a1}\varsigma_{a12}-\eta_{a2}\varsigma_{a21}=\\&=\left(\eta_{a1}(1+\zeta_{a12})-\eta_{a2}(1+\zeta_{a21})\right)/2\\&+\cos 2(\phi-\theta_a)\left(\eta_{a1}(1-\zeta_{a12})+\eta_{a2}(1-\zeta_{a21})\right)/2,\end{aligned}\qquad(14)$$

the total probability of registration by Alice is

$$\begin{aligned}N(\mathbf{a},\boldsymbol{\gamma})&=\eta_{a1}(1-\zeta_{a12})\cos^2(\phi-\theta_a)\\&+\eta_{a2}(1-\zeta_{a21})\sin^2(\phi-\theta_a)+\eta_{a1}\varsigma_{a12}+\eta_{a2}\varsigma_{a21}=\\&=\left(\eta_{a1}(1+\zeta_{a12})+\eta_{a2}(1+\zeta_{a21})\right)/2\\&+\cos 2(\phi-\theta_a)\left(\eta_{a1}(1-\zeta_{a12})-\eta_{a2}(1-\zeta_{a21})\right)/2,\end{aligned}\qquad(15)$$

correlation probability for the Bob's photon to be

$$\begin{aligned}P(\mathbf{b},\boldsymbol{\gamma})&=\left(\eta_{b1}(1+\zeta_{b12})-\eta_{b2}(1+\zeta_{b21})\right)/2\\&+\cos 2(\phi-\theta_b)\left(\eta_{b1}(1-\zeta_{b12})+\eta_{b2}(1-\zeta_{b21})\right)/2,\end{aligned}\qquad(16)$$

and the total probability of his registration is

$$\begin{aligned}N(\mathbf{b},\boldsymbol{\gamma})&=\left(\eta_{b1}(1+\zeta_{b12})+\eta_{b2}(1+\zeta_{b21})\right)/2\\&+\cos 2(\phi-\theta_b)\left(\eta_{b1}(1-\zeta_{b12})-\eta_{b2}(1-\zeta_{b21})\right)/2.\end{aligned}\qquad(17)$$

From that one can immediately put down the correlation function of Alice and Bob measurements

$$E(\mathbf{a},\mathbf{b})=\frac{\int d\varphi P(\mathbf{a},\boldsymbol{\gamma})P(\mathbf{b},\boldsymbol{\gamma})}{\int d\varphi N(\mathbf{a},\boldsymbol{\gamma})N(\mathbf{b},\boldsymbol{\gamma})}=\frac{C+(D/2)\cos 2\theta_{ab}}{F+(G/2)\cos 2\theta_{ab}},\qquad(18)$$

where

$$C=\left(\eta_{a1}(1+\zeta_{a12})-\eta_{a2}(1+\zeta_{a21})\right)\left(\eta_{b1}(1+\zeta_{b12})-\eta_{b2}(1+\zeta_{b21})\right),\qquad(19$$

$$D=\left(\eta_{a1}(1-\zeta_{a12})+\eta_{a2}(1-\zeta_{a21})\right)\left(\eta_{b1}(1-\zeta_{b12})+\eta_{b2}(1-\zeta_{b21})\right),\qquad(20)$$

$$F=\left(\eta_{a1}(1+\zeta_{a12})+\eta_{a2}(1+\zeta_{a21})\right)\left(\eta_{b1}(1+\zeta_{b12})+\eta_{b2}(1+\zeta_{b21})\right),\qquad(21)$$

$$G=\left(\eta_{a1}(1-\zeta_{a12})-\eta_{a2}(1-\zeta_{a21})\right)\left(\eta_{b1}(1-\zeta_{b12})-\eta_{b2}(1-\zeta_{b21})\right).\qquad(22)$$

It is easy to check that, if all ζ=0, and all η=1, then (18) becomes identical to (13).

In the case of entangled wave function (1) one can derive

$$E(\mathbf{a},\mathbf{b})=\frac{C+D\cos 2\theta_{ab}}{F+G\cos 2\theta_{ab}}\qquad(23)$$

with the same coefficients (19)-(22). If all ζ=0, but all η≠1 and are different, then correlation functions are representable as

$$E(\mathbf{a},\mathbf{b})=\frac{\alpha+x}{1+\alpha x},\qquad(24)$$

where α=C/D, and x= $\cos 2\theta_{ab}$ or $\cos 2\theta_{ab}$ /2. From this expression one can derive an inequality

$$|x|=\left|\frac{E(\mathbf{a},\mathbf{b})-\alpha}{1-\alpha E(\mathbf{a},\mathbf{b})}\right|\le\frac{1}{2}\qquad(25)$$

for the individual photons model.

Of course, it is also necessary to take into account statistical uncertainties related to dispersions of photon counts and subtraction of background. They are not considered here, because, it seems, that presented considerations are quite enough to show that even without statistics a careful check of (25) represents a formidable task already in a single arrangement of two calcite axes, so the four arrangements lead to at least four times more difficulties.

It is important also to notice that measured correlations are very sensitive to subtraction of the. Too high background always increases the correlation, and in this way it is easy to demonstrate violation of the Bell's inequalities.

6. Conclusion

From the above it follows that for a final check of two versions of quantum mechanics with individual or entangled photons it is required an experiment with parallel aligned crystals and rare pulses of the photon pairs, where each pair is marked by the time of registration, and Alice takes measurements the first and accepts only photons registered, say, by the detector D_{1A}. Only these photon pairs are accepted by Bob, whose task is to check, whether some of these photons are registered by his detector D_{2B}. In this experiment, there will be no loophole, and one of the types of quantum mechanics will be completely ruled out.

References

[1] Bohm D, Aharonov Y. 1957. Discussion of experimental proof for the paradox Einstein, Rosen, and Podolsky. Phys.Rev.108: 1070-6.

[2] Bell J.S. 1964. Physics. 1: 195.

[3] Bell J.S. 2004. Speakable and unspeakable in quantum mechanics. Cambridge University Press, p. 14.

[4] Clauser J. F., M. A. Horn, A. Shimony, R. A. Holt, 1969. Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. 23: 880.

[5] Aspect A, P. Grangier, G. Roger. 1981. Experimental tests of realistic local theories via Bell's theorem. Phys. Rev. Lett. 47: 460-3.

[6] Hayashi M. et all. 2006. Hypothesis testing for an entangled state produced by spontaneous parametric down-conversion, Phys. Rev. A 74: 062321.

[7] Poh H.S., Siddarth K. Joshi, Alessandro Cerè, Adán Cabello, and Christian Kurtsiefer. 2015. Approaching Tsirelson's Bound in a Photon Pair Experiment. Phys. Rev. Lett 115: 180408.

[8] Ignatovich V. K. 2008. On EPR paradox, Bell's inequalities and experiments that prove nothing. Concepts of Physics, the old and new. 5: 227.

[9] Ignatovich V.K. 2015. Closer Look at EPR Paradox and Bell's Inequality. American Journal of Modern Physics and Application, 2(2): 16-20.

[10] Laloe F. 2001. Do we really understand quantum mechanics? Strange correlations, paradoxes, and theorems. Am. J. Phys. 69: 655-701.

[11] Norsen T. 2011. John S. Bell's concept of local causality. Am. J. Phys. 79: 1261-1275.