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Generation of Static Perfect Fluid Spheres in General Relativity

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Abstract

In this paper a new technique of finding static spherically symmetric perfect fluid solutions is presented. This amounts to solving two first order differential equations, one for each of the two metric functions, coupled by a single generating function. The technique can be applied to generate new solutions from previously known solutions. Using the technique a new physically acceptable solution is generated.

Keywords

Isotropy, Anisotropy, Metric Functions, Space-time, Perfect Fluid, Central Pressure, Central Density

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1. Introduction

Spherically symmetric solutions have attracted attentions of many researchers working in this field due to several reasons. Spherically symmetric perfect fluid solutions are interesting because they are first approximations in finding any realistic solution [3-11] describing a relativistic star. The first such solution with constant density was found by Schwarzschild [3] in 1918. Since then a number of static spherically symmetric perfect fluid solutions have been discovered. At this stage it seems difficult to find new solutions by directly solving Einstein's equation. In recent years some solution generating techniques for generating new solutions without directly solving Einstein's equation have been discovered. In [1-2] some solution generating techniques are presented which provide a way to classify the set of perfect fluid solutions systematically. Moreover these techniques can be used to generate new solutions from known solutions. These techniques are based on the pressure isotropy condition but not of anisotropy condition, following Buchdahl [12] limit.

In Section-2, we derive the field equations for static spherically symmetric perfect fluid source in curvature coordinates and caste these equations in a form convenient for our purpose. In Section-3, a brief review of the solution generating techniques is provided. In Section-4, we present a new technique for generating solutions from known solutions. Using the technique a class of new realistic solutions is generated in Section-5. In Section-6, we discuss on properties of the solution. Finally in Section-7, some concluding remarks are given.

2. Field Equations

The metric of static spherically symmetric space-time in curvature coordinates can be written as

$$\mathrm{ds}^2 = -\mathrm{e}^{2\phi(\mathrm{r})}\mathrm{dt}^2 + \mathrm{e}^{2\Lambda(\mathrm{r})}\mathrm{dr}^2 + \mathrm{r}^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \quad (1)$$

For the metric (1) the nonzero components of Einstein's tensor are given by

$$G_{00} = \frac{1}{r^2} e^{2\phi} \frac{d}{dr}\left\{ r\left(1 - e^{-2\Lambda}\right)\right\} \quad (2a)$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda}\left(1 - e^{-2\Lambda}\right) + \frac{2\phi'}{r} \quad (2b)$$

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$$G_{\theta\theta} = r^2 e^{-2\Lambda}\left\{\phi'' + (\phi')^2 + \frac{\phi'}{r} - \left(\phi' + \frac{1}{r}\right)\Lambda'\right\} \quad (2c)$$

$$G_{\phi\phi} = G_{\theta\theta}\sin^2\theta$$

where primes denote derivatives with respect to r. We consider matter content of the fluid sphere to be perfect fluid. In this case nonzero components of the energy-momentum tensor $T_{\mu\nu}$ are given by

$$T_{00} = \rho e^{2\phi} \quad (3a)$$

$$T_{rr} = p e^{2\Lambda} \quad (3b)$$

$$T_{\theta\theta} = pr^2 \quad (3c)$$

$$T_{\phi\phi} = T_{\theta\theta}\sin^2\theta$$

where $\rho(r)$ and p(r) are respectively the density and pressure within the fluid sphere. Putting (2) and (3) in the Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ we obtain the independent equations

$$8\pi\rho = \frac{1}{r^2}e^{2\phi}\frac{d}{dr}\left\{r\left(1-e^{-2\Lambda}\right)\right\} \quad (4a)$$

$$8\pi p = -\frac{1}{r^2}\left(1-e^{-2\Lambda}\right) + \frac{2}{r}e^{-2\Lambda}\phi' \quad (4b)$$

$$8\pi p = r^2 e^{-2\Lambda}\left\{\phi'' + (\phi')^2 + \frac{\phi'}{r} - \left(\phi' + \frac{1}{r}\right)\Lambda'\right\} \quad (4c)$$

From (4b) and (4c) we obtain the equation

$$\frac{d}{dr}\left\{e^{-2\Lambda}\right\} + \frac{2\left\{r^2\phi'' + r^2(\phi')^2 - r\phi' - 1\right\}}{r(r\phi'+1)}e^{-2\Lambda} = -\frac{2}{r(r\phi'+1)} \quad (5)$$

If we let $e^{-2\Lambda} = 1 - \frac{2m}{r}$ and $u = \frac{m}{r}$, then equation (5) reduces to

$$u' + \frac{2\left\{r^2\phi'' + (r\phi')^2 - r\phi' - 1\right\}}{r(r\phi'+1)}u = \frac{r^2\phi'' + (r\phi')^2 - r\phi'}{r(r\phi'+1)} \quad (6)$$

Putting $e^{-2\Lambda} = 1 - \frac{2m}{r}$ and $u = \frac{m}{r}$ in (4a) and (4b) we get

$$4\pi\rho = \frac{m'(r)}{r^2} \quad (7)$$

$$4\pi p = \frac{r\phi'(1-2u) - u}{r^2} \quad (8)$$

Equation (6) restricts the freedom of choosing the two metric functions $\phi(r)$ and m(r) to one. This is a first order linear differential equation in u and can be solved if $\phi(r)$ is specified. Knowing u(r) and $\phi(r)$ density and pressure can be determined by putting these in (7) and (8).

3. Solution Generating Techniques

In [1] and [2] several solutions generating techniques are presented. Using these techniques a known solution can be transformed into a new solution. To see how this can be done let $N(r) = e^{\phi(r)}$ and $G(r) = e^{-\Lambda(r)}$. Then equation (6) reduces to

$$G' + \frac{2(r^2N'' - rN' - N)}{r(rN' + N)}G + \frac{2N}{r(rN'+N)} = 0 \quad (9)$$

Equation (9) can be regrouped as

$$(2r^2G)N'' + (r^2G' - 2rG)N' + (rG' - 2G + 2)N = 0 \quad (10)$$

Solution generating techniques can be described as follows:

(i) Suppose we start with a solution $\left(N_0(r), G_0(r)\right)$ i.e. we suppose that the equation

$$G_0' + \frac{2(r^2N_0'' - rN_0' - N_0)}{r(rN_0' + N_0)}G_0 + \frac{2N_0}{r(rN_0' + N_0)} = 0 \quad (11)$$

is satisfied. Let us now demand that $\left(N_0(r), G_1(r)\right)$, where $G_1(r) = G_0(r) + k\Delta_0(r)$ is a solution. Putting $\left(N(r), G(r)\right)$ = $\left(N_0(r), G_1(r)\right)$ in (9) and using (11) we get

$$\Delta_0' + \frac{2(r^2N_0'' - rN_0' - N_0)}{r(rN_0' + N_0)}\Delta_0 = 0 \quad (12)$$

Solution of (12) is given by

$$\Delta_0(r) = \frac{r^2}{\{(rN_0)'\}^2}\exp\{\int\frac{4N_0'}{(rN_0)'}dr\} \quad (13)$$

Therefore we conclude that if $\left(N_0, G_0\right)$ is a known solution and if $\Delta_0(r)$ is given by (13) then $\left(N_0, G_0(r) + k\Delta_0(r)\right)$ is also a solution.

The result can be viewed as a transformation

$$T_1 : \left(N_0, G_0\right) \rightarrow \left(N_0, G_0(r) + k\Delta_0(r)\right)$$

in the set of solutions of equation (9).

(ii) Let (N_0, G_0) be a solution of (10) i.e. we let

$$(2r^2G_0)N_0'' + (r^2G_0' - 2rG_0)N_0' + (rG_0' - 2G_0 + 2)N_0 = 0 \quad (14)$$

and demand that $(N_1(r), G_0(r))$, where $N_1(r) = N_0(r)Z(r)$, be a solution. Putting (N_1, G_0) in (10) and using (14) we get the equation

$$Z_0'' + \frac{r^2N_0G_0' - 4r^2N_0'G_0 - 2rN_0G_0}{2r^2N_0G_0} Z_0' = 0 \quad (15)$$

which is a first order linear differential equation in Z_0'. Solution of (15) is given by

$$Z_0 = c_1 + c_2 \int \frac{rdr}{N_0^2\sqrt{G_0}} \quad (16)$$

which depend on the starting solution (N_0, G_0). Therefore we conclude that if (N_0, G_0) is a solution then (N_0Z_0, G_0), where Z_0 is given by (16), is also a solution. The result can be viewed as the transformation

$$T_2 : (N_0, G_0) \to (N_0Z_0(N_0, G_0), G_0)$$

in the set of solutions.

(iii) The composite transformation T_3 defined by

$$T_3(N_0, G_0) = T_2(T_1(N_0, G_0))$$

generates the solution $(N_0Z_0(N_0, G_0 + k\Delta_0), G_0 + k\Delta_0)$ from the known solution (N_0, G_0),

$$T_3 : (N_0, G_0) \to (N_0Z_0(N_0, G_0 + k\Delta_0), G_0 + k\Delta_0)$$

where $\Delta_0 = \Delta_0(N_0)$ i.e. Δ_0 depends on N_0.

(iv) It can be seen that, in general $T_2 \circ T_1 \neq T_1 \circ T_2$. The composite transformation defined by

$$T_4(N_0, G_0) = T_1(T_2(N_0, G_0))$$

generates the solution $(N_0Z_0(N_0, G_0), G_0 + k\Delta_0(N_0Z_0(N_0, G_0))$ from the known solution (N_0, G_0),

$$T_4 : (N_0, G_0) \to (N_0Z_0(N_0, G_0), G_0 + k\Delta_0(N_0Z_0(N_0, G_0))$$

It should be noted that, the transformations T_1, T_2, T_3 or T_4 when applied to a metric (N_0, G_0), the output metric (N, G) may or may not be distinct from the input metric (N_0, G_0). A metric (N_0, G_0) for which the transformations T_1 and T_2 both yield output metrics distinct from the input metric is termed a seed metric. In contrast for nonseed metrics one or the other of the two transformations T_1 and T_2 yield a metric which is not distinct from the input metric.

4. A New Technique for Generation of Perfect Fluid Spheres

Equation (6) can be regrouped as follows,

$$\phi'' = \frac{ru' - 2u}{r^2(1-2u)} + \frac{ru' - 2u + 1}{r(1-2u)}\phi' - (\phi')^2 \quad (17)$$

Equation (17) is a Riccati type differential equation of type

$$v'(r) = P(r) + Q(r)v(r) - v^2(r) \quad (18)$$

where $v(r) = \phi'(r)$, $P(r) = \frac{ru' - 2u}{r^2(1-2u)}$ and

$$Q(r) = \frac{ru' - 2u + 1}{r(1-2u)} \quad (19)$$

We know that, if one particular solution of a Riccati type differential equation is known then its general solution can be obtained using standard procedures.

Let (u_0, v_0) is a known solution of (6) and hence of (18). Then the following equation is satisfied,

$$v_0' = P_0(r) + Q_0(r)v_0 - v_0^2 \quad (20)$$

where $P_0(r) = \frac{ru_0' - 2u_0}{r^2(1-2u_0)}$ and $Q_0(r) = \frac{ru_0' - 2u_0 + 1}{r(1-2u_0)}$

From (20) we find that $v(r) = v_0$ is a particular solution of the Riccati differential equation

$$v' = P_0(r) + Q_0(r)v - v^2 \quad (21)$$

General solution of (21) is given by

$$V = v_0 + \frac{1}{z} \quad (22)$$

where z(r) is the solution of the first order linear equation

$$z' + (Q - 2v_0)z = 1 \quad (23)$$

Then we have

$$V' = P_0(r) + Q_0(r)V - V^2 \tag{24}$$

Equation (24) can be regrouped as follows,

$$u_0' + \frac{2(r^2V' + r^2V^2 - rV - 1)}{r(rV+1)}u_0 = \frac{r^2V' + r^2V^2 - rV}{r(rV+1)} \tag{25}$$

Hence u = u_0 is a solution of the differential equation

$$u' + \frac{2(r^2V' + r^2V^2 - rV - 1)}{r(rV+1)}u = \frac{r^2V' + r^2V^2 - rV}{r(rV+1)} \tag{26}$$

Now let us demand that u(r) = u_0(r) + U(r) is a solution of (26). Putting u = u_0 + U in (26) and using (25) we obtain the equation

$$U' + \frac{2(r^2V' + r^2V^2 - rV - 1)}{r(rV+1)}U = 0 \tag{27}$$

Therefore we have shown that, if (u_0,v_0) is a static spherically symmetric perfect fluid solution and if z(r) and U(r) are solutions of (23) and (27) respectively then $(u_0 + U, v_0 + \frac{1}{z})$ is also a static spherically symmetric perfect fluid solution.

In the next section we have found a class of new static spherically symmetric perfect fluid solutions using the solution generating technique described above.

5. New Solutions

Let us start with the solution $(u_0, v_0) = (-\frac{ar^2}{2}, 0)$.

From the third of equation (19) we obtain

$$Q_0(r) = \frac{1}{r(1+ar^2)} \tag{28}$$

Putting (28) and v_0 = 0 in (23) we get

$$z' + \frac{1}{r(1+ar^2)}z = 1 \tag{29}$$

Solution of (29) is found to be

$$z = \frac{1 + ar^2 - l\sqrt{1+ar^2}}{ar} \tag{30}$$

where l is an arbitrary constant. General solution of (21) is obtained by putting (30) in (22) and is given

$$V = v_0 + \frac{1}{z} = \frac{ar}{1 + ar^2 - l\sqrt{1+ar^2}} \tag{31}$$

Putting (31) in (27) we obtain

$$\frac{U'}{U} + \frac{2ar}{x^2(2x^2 - lx - 1)} - \frac{2}{r} = 0 \tag{32}$$

where $x = \sqrt{1+ar^2}$. Solution of (32) is found to be

$$U(r) = \frac{cr^2x^2}{2x^2 - lx - 1}\left(\frac{4x - l - \sqrt{l^2+8}}{4x - l + \sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}} \tag{33}$$

where c is an arbitrary constant. Therefore

$$u = u_0 + U = -\frac{ar^2}{2} + \frac{cr^2x^2}{2x^2 - lx - 1}\left(\frac{4x - l - \sqrt{l^2+8}}{4x - l + \sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}} \tag{34}$$

Since $V = \phi'$ from (31) we obtain

$$\phi(r) = \int \frac{ardr}{1 + ar^2 - l\sqrt{1+ar^2}}$$

This gives

$$e^{2\phi} = A^2(-l + \sqrt{1+ar^2})^2 \tag{35}$$

$$\text{Also } e^{-2\Lambda} = 1 - \frac{2m}{r} = 1 - 2u \tag{36}$$

Putting (34) into (36) we obtain

$$e^{-2\Lambda} = 1 + ar^2 - \frac{cr^2x^2}{2x^2 - lx - 1}\left(\frac{4x - l - \sqrt{l^2+8}}{4x - l + \sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}} \tag{37}$$

Putting (35) and (37) in (1) we obtain the following metric as our new solution,

$$ds^2 = -A^2(-l + \sqrt{1+ar^2})^2dt^2 + \frac{dr^2}{1 + ar^2 - \frac{cr^2x^2}{2x^2 - lx - 1}\left(\frac{4x - l - \sqrt{l^2+8}}{4x - l + \sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}}}$$

$$+ r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{38}$$

Metric (38) describes a one parameter family of static spherically symmetric perfect fluid solutions parameterized by c. It reduces to Schwarzschild constant density solution if c = 0.

6. Properties of the Solution

Density and pressure can be found by putting (37), [with $e^{-2\Lambda} = 1 - \frac{2m}{r}$], and (31), [with $V = \phi'$], in (7) and (8). We get

$$\rho(\mathrm{r})=\frac{1}{8\pi}\left[\frac{1}{2\mathrm{x}^2-l\mathrm{x}-1}\left(\frac{4\mathrm{x}-l-\sqrt{l^2+8}}{4\mathrm{x}-l+\sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}}\left(3\mathrm{cx}^2-\frac{2\mathrm{acr}^2}{2\mathrm{x}^2-l\mathrm{x}-1}\right)-3a\right] \quad (39)$$

$$p(\mathrm{r})=\frac{1}{8\pi}\left[\frac{(3x-l)a}{x-l}-\frac{\mathrm{cx}}{2\mathrm{x}^2-l\mathrm{x}-1}\left(\frac{4\mathrm{x}-l-\sqrt{l^2+8}}{4\mathrm{x}-l+\sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}}\left(\frac{3\mathrm{x}^2-l\mathrm{x}-2}{x-l}\right)\right] \quad (40)$$

Central density and central pressure can be obtained by putting r = 0 i.e. x = 1, in (39) and (40). These are found to be given by

$$\rho_c=\frac{1}{8\pi}\left[\frac{3\mathrm{c}}{1-l}\left(\frac{4-l-\sqrt{l^2+8}}{4-l+\sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}}-3a\right] \quad (41)$$

$$p_c=\frac{1}{8\pi}\left[\frac{(3-l)a}{1-l}-\frac{\mathrm{c}}{1-l}\left(\frac{4-l-\sqrt{l^2+8}}{4-l+\sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}}\right] \quad (42)$$

Let r = R be the surface of the fluid sphere so that p(R) = 0. From (40) we then get

$$\frac{c}{a}=\frac{(3x_0-l)(2x_0^2-lx_0-1)}{x_0(3x_0^2-lx_0-2)}\left(\frac{4x_0-l+\sqrt{l^2+8}}{4x_0-l-\sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}} \quad (43)$$

where $x_0=\sqrt{aR^2+1}$. Since $u=\frac{m}{r}$ from (34) we obtain

$$m(r)=-\frac{ar^3}{2}+\frac{cr^3x^2}{2x^2-lx-1}\left(\frac{4x-l-\sqrt{l^2+8}}{4x-l+\sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}} \quad (44)$$

Mass M = m(R) of the fluid sphere can be obtained by putting r = R and $x_0=\sqrt{aR^2+1}$ in (44). We get

$$M=-\frac{aR^3}{2}+\frac{cR^3x_0^2}{2x_0^2-lx_0-1}\left(\frac{4x_0-l-\sqrt{l^2+8}}{4x_0-l+\sqrt{l^2+8}}\right)^{\frac{l}{\sqrt{l^2+8}}} \quad (45)$$

Since all interior solutions must be joined smoothly onto the vacuum Schwarzschild solution we must have

$$e^{2\phi(R)}=1-\frac{2M}{R}=A^2\left(-l+\sqrt{1+aR^2}\right)^2 \quad (46)$$

This gives
$$A=\frac{\sqrt{1-\frac{2M}{R}}}{-l+\sqrt{1+aR^2}} \quad (47)$$

Putting (47) in (35) we obtain

$$e^{2\phi(r)}=\left(1-\frac{2M}{R}\right)\left(\frac{-l+\sqrt{1+ar^2}}{-l+\sqrt{1+aR^2}}\right)^2,\ r\le R \quad (48)$$

SPECIFIC CASE $l=-1$

Putting $l=-1$ in equations (39) – (48) we obtain

$$\rho(\mathrm{r})=\frac{2^{\frac{1}{3}}}{8\pi}\left[\frac{3\mathrm{cx}^2}{(\mathrm{x}+1)^{\frac{2}{3}}(2x-1)^{\frac{4}{3}}}-\frac{2\mathrm{acr}^2}{(\mathrm{x}+1)^{\frac{5}{3}}(2x-1)^{\frac{7}{3}}}-3a\right] \quad (49)$$

$$p(\mathrm{r})=\frac{1}{8\pi}\left[\frac{(3\mathrm{x}+1)\mathrm{a}}{\mathrm{x}+1}-\frac{2^{\frac{1}{3}}\mathrm{cx}(3\mathrm{x}-2)}{(\mathrm{x}+1)^{\frac{2}{3}}(2x-1)^{\frac{4}{3}}}\right] \quad (50)$$

$$\rho_c=\frac{2^{\frac{1}{3}}}{8\pi}\left(\frac{3\mathrm{c}}{2^{\frac{2}{3}}}-3a\right) \quad (51)$$

$$p_c=\frac{1}{8\pi}\left(2a-\frac{\mathrm{c}}{2^{\frac{1}{3}}}\right) \quad (52)$$

$$\frac{c}{a}=\frac{(3x_0+1)(2x_0-1)^{\frac{4}{3}}}{2^{\frac{1}{3}}x_0(x_0+1)^{\frac{1}{3}}(3x_0-2)} \quad (53)$$

$$m(r)=-\frac{ar^3}{2}+\frac{\mathrm{cr}^3\mathrm{x}^2}{2^{\frac{1}{3}}(\mathrm{x}+1)^{\frac{4}{3}}(2x-1)^{\frac{2}{3}}} \quad (54)$$

$$M=-\frac{aR^3}{2}+\frac{\mathrm{cR}^3\mathrm{x}_0^2}{2^{\frac{1}{3}}(\mathrm{x}_0+1)^{\frac{4}{3}}(2x_0-1)^{\frac{2}{3}}} \quad (55)$$

$$e^{2\phi(R)}=1-\frac{2M}{R}=A^2\left(1+\sqrt{1+aR^2}\right)^2 \quad (56)$$

$$A=\frac{\sqrt{1-\frac{2M}{R}}}{1+\sqrt{1+aR^2}} \quad (57)$$

$$e^{2\phi(r)}=\left(1-\frac{2M}{R}\right)\left(\frac{1+\sqrt{1+ar^2}}{1+\sqrt{1+aR^2}}\right)^2,\ r\le R \quad (58)$$

From (55) we get

$$\frac{M}{R}=\frac{aR^2}{3aR^2+\sqrt{aR^2+1}+1}=\frac{1}{k},\text{say} \quad (59)$$

From (59) we get

$$x_0=1+\frac{1}{k-3} \quad (60)$$

Since $x_0 > 1$ from (60) we find that $k > 3$. From (59) we get

$$\frac{M}{R} < \frac{1}{3} < \frac{4}{9}$$

Hence Buchdahl condition holds. From (51) and (52) we find that, $\rho_c > 0$ and $p_c > 0$ requires $2^{\frac{2}{3}} < \frac{c}{a} < 2^{\frac{4}{3}}$. From (53) it can be seen that this condition is satisfied for $x_0 = 2$, for example.

7. Conclusion

We have presented a technique based on the choice of a generating function which can generate all static spherically symmetric perfect fluid solutions of Einstein equations. The technique can also be used to generate new solutions from known solutions. However not all solutions generated thus are physically acceptable. For obtaining physically acceptable solutions the generating function must satisfy some conditions. In spite of this reservation we have been able to construct a physically acceptable solution.

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