**Exact Longitudinal Natural Frequencies of Single-Walled Carbon Nanotubes via Eringen’s Nonlocal Theory and the Transfer Matrix Method**

**Vebil Yıldırım***

Mechanical Engineering Department, University of Çukurova, Adana, Turkey

**Abstract**

The principal purpose of the present study is to introduce the transfer matrix method, which is an efficacious and accurate analytical/numerical method developed based on the initial value problem (IVP), to analytical evaluation of the longitudinal natural frequencies of a single-walled carbon nanotube (SWCNT) based on Eringen’s nonlocal elasticity theory under classical boundary conditions. After validation of the results, it is graphically shown that there is an upper bound (cut-off/corner/break frequency) for the non-local natural frequencies contrary to the local ones. The numerical value of this bound depends on the geometrical and material properties together with the boundary conditions. The effects of vibrational parameters on the natural frequencies are also investigated.

**Keywords**

Transfer Matrix Approach, Exact Solution, Initial Value Problem, Free Longitudinal Vibration, Nonlocal Elasticity Theory

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1. Introduction

As a new form of carbon, carbon nanotubes (CNTs) discovered in early 1990s are one of the stiffest and strongest structural elements that are widely used in miscellaneous nano-sized and micro-sized mechanical, biomedical, chemical and biological applications such as micro beams, microfilms, biosensors, nanowires, atomic force microscopes, nanotubes, tear-resistant cloth fibers, micro actuators, nano-probes, lighter and stronger sports equipment, micro-electro-mechanical systems (MEMS), flat screen displays, ultra-thin films, and so on. They are projected as hollow cylindrical bars of graphitic carbon at the nano-scale which are configurationally equivalent to two dimensional graphene sheet rolled into a seamless cylinder consisting of a hexagonal network of carbon atoms (Figure 1). They may be produced with a diameter of a few nanometers and a length of several microns. Single-walled carbon nanotubes (SWCNTs) and multi-walled carbon nanotubes (MWCNTs) consisting of multiple layers of graphite rolled in on themselves to form a tube shape are the main forms of carbon nanotubes.

* Corresponding author
E-mail address: vebil@cu.edu.tr

![Figure 1. A single-walled carbon nanotube (SWCNT) [1].](image-url)
properties [2-6], carbon nanotubes continue to draw heavy academic and industrial attention, as expected.

The classical elasticity theory (local elasticity theory) only pay attention to the macroscopic effects. However, the length scale parameters play a key role on the mechanical behavior of nano/micro-structures [2-6]. As a result, mechanical properties of materials at the nano/micro-scale differ significantly from those of their bulk counterparts. Those effects may be considered by employing appropriate one of the higher-order continuum theories instead of atomistic simulation models or hybrid atomistic-continuum models which are both computationally expensive methods.

The non-local elasticity theory proposed by Eringen [7-8] is one of the higher-order continuum theories such as Cosserat elasticity, strain gradient elasticity by Mindlin, micropolar theory by Eringen and Suhubi, couple stress theory of elasticity by Mindlin and Tiersten, by Toupin, and by Koiter, simple gradient elasticity with surface energy by Vardoulakis and Sulem, and by Altan and Aifantis. Zhang et al. [9-10] firstly employed modeling and mechanical analysis of carbon nanotubes via nonlocal elasticity theory. Reddy [11] investigated bending, buckling and free vibration analysis of nonlocal beams regarding to the different beam theories.

The present study only deals with the axial vibrations of single-walled carbon nanotubes [12-22]. From those Altan et al. [12] examined the longitudinal vibrations of a beam based on a gradient elasticity approach. Aydogdu [13] originally investigated the small-scale effect on longitudinal vibration of a carbon nanotube based on Eringen’s nonlocal elasticity theory. Filiz and Aydogdu [14] considered axial vibration of carbon nanotube heterojunctions using nonlocal elasticity theory. By employing a perturbation technique, Kiani [15] studied free longitudinal vibration of tapered nanowires based on the nonlocal continuum theory. Murmu and Adhikari [16] worked on the axial vibration of double-nanorod systems. Based on nonlocal elasticity theory and differential quadrature method, Danesh et al. [17] performed an axial vibration analysis of a tapered nanorod. Simsek [18] studied the free longitudinal vibration of axially functionally graded tapered nanorods. Akgöz and Civalek [19] originally formulated the longitudinal free vibration problem of a micro-scaled bar by using the strain gradient elasticity theory proposed by Lam et al. [23]. Akgöz and Civalek [19] showed that size effect is more significant when the ratio of the microbar diameter to the additional length scale parameter is small. They also revealed that the difference between natural frequencies predicted by current and classical models becomes more prominent for both lower values of slenderness ratio of the microbar and for higher modes. Li et al. [20] used nonlocal theoretical approaches and atomistic simulations for longitudinal free vibration of nanorods/nanotubes. Li et al. [21] employed a nonlocal strain gradient theory to study the longitudinal vibration of size-dependent rods based on the finite element method under classical and non-classical boundary conditions. Li et al.’s [21] model contains a nonlocal parameter considering the significance of nonlocal elastic stress field and a material length scale parameter considering the significance of strain gradient stress field. Li et al. [21] showed that the nonlocal strain gradient rod model exerts a stiffness-softening effect when the nonlocal parameter is larger than the material length scale parameter, and exerts a stiffness-hardening effect when the nonlocal parameter is smaller than the material length scale parameter. Li et al. [21] also revealed that the higher-order frequencies are more sensitive to the non-classical boundary conditions in comparison with the lower-order frequencies, and the type of non-classical boundary conditions has a little effect on mode shapes. Xu et al. [22], recently, investigated the size effects on the longitudinal vibrational behavior of rods within the framework of the nonlocal strain gradient elastic theory. Similar to the fourth-order differential equation of motion for classical Euler–Bernoulli beams, Xu et al. [22] clarified for the first time the variationally consistent boundary conditions of nonlocal strain gradient rods. They presented axial natural frequencies for four types of boundary value problems. Xu et al. [22] showed that both the softening effect and the stiffening effect can be captured by adjusting the two material length parameters.

Artan and Batra [24] originally employed the transfer matrix method to find the bending frequencies of free vibrations of a strain-gradient-dependent Euler–Bernoulli beam. This method, now, is to be originally applied to the free longitudinal vibration analysis of SWCNTs based on Eringen’s nonlocal elasticity theory in the present study [8].

2. Formulation and Solution

Eringen [8] offered the following nonlocal constitutive differential equation

\[
\left(1 - (e_v a)^2 \right) \mu_v \sigma \cdot \varepsilon = C \cdot \varepsilon = C_{ij} \varepsilon_{ij} \mu
\]  (1)

where \( a \) is the internal characteristic length, \( e_v \) is a constant, \( \mu = e_o a < 2nm \) is the scale factor or scale coefficient of a single-walled carbon nanotube (SWCNT), the colon, :, denotes the double dot product, \( \nabla^2 \) is the Laplacian operator, \( \sigma \) is the stress tensor, \( C \) and \( \varepsilon \) are the elasticity and strain tensors, respectively. For a given material, numerical value of \( \mu \) is determined based on the atomic lattice dynamics and experiments.

If one-dimensional longitudinal motion of a uniform rod
along the x-axis is concerned (Figure 2), the non-local constitutive equation in Eq. (1) turns to be

\[ 1 - \mu^2 \frac{d^2}{dx^2} \sigma_{NL}^{(x,t)} = E \varepsilon(x,t) \]  

(2)

where \( E \) is Young’s modulus of the beam material, and \( \varepsilon \) is the axial unit strain. The local elasticity theory defines the axial force \( N(x,t) \) as follows

\[ N(x,t) = \int_A \sigma_{NL}^{(x,t)}(x,t) dA = \int_A E \varepsilon(x,t) dA \]  

(3)

Integration of Eq. (2) over the un-deformed cross sectional area, \( A \), gives

\[ \int_A \left(1 - \mu^2 \frac{d^2}{dx^2} \right) \sigma_{NL}^{(x,t)} dA = \int_A E \varepsilon(x,t) dA \]  

(4)

This equation may be written as follows

\[ N(x,t) = \mu \frac{d^2 N(x,t)}{dx^2} = N'(x,t) \]  

(5)

or by using the following

\[ N'(x,t) = EA \frac{du(x,t)}{dx} \]  

(6)

it takes the following form.

\[ N_{NL}^{(x,t)} = \mu^2 \frac{d^2 N(x,t)}{dx^2} + EA \frac{du(x,t)}{dx} \]  

(7)

In eqs. (6) and (7), \( u(x,t) \) stands for the longitudinal displacement. From the equation of motion of an infinitesimal element of a uniform rod the following may be obtained from the local elasticity theory.

\[ \frac{\partial N'(x,t)}{\partial x} + f(x,t) = \rho A \frac{d^2 u(x,t)}{dt^2} \]  

(8)

where \( t \) is the time, \( \rho \) is the material density, and \( f(x) \) is the distributed axial forces. It is worth noting that Eq. (8) should be satisfied in both local and non-local elasticity theories. The derivative of Eq. (8) with respect to \( x \), gives

\[ \frac{\partial^2 N_{NL}^{(x,t)}}{\partial x^2} = - \frac{\partial f(x,t)}{\partial x} + \rho A \frac{\partial}{\partial x} \left( \frac{\partial^2 u(x,t)}{\partial t^2} \right) \]  

(9)

Substitution of Eq. (9) into Eq. (7) yields the non-local axial force as follows.

\[ N_{NL}^{(x,t)} = \mu^2 \left( \rho A \frac{\partial}{\partial x} \left( \frac{\partial^2 u(x,t)}{\partial t^2} \right) - \frac{\partial f(x,t)}{\partial x} \right) + EA \frac{\partial u(x,t)}{\partial x} \]  

(10)

With the help of Eqs. (8) and (10), the equilibrium equation may, now, be written based on the nonlocal elasticity theory as

\[ \frac{\partial}{\partial x} \left( \mu^2 \left( \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 f(x,t)}{\partial x^2} \right) \right) + \rho A \frac{\partial^2 u(x,t)}{\partial t^2} \]  

(11)

or

\[ \frac{\partial}{\partial x} \left( \mu \frac{d^2 u(x,t)}{dx^2} \right) + f(x,t) = \rho A \frac{\partial^2 u(x,t)}{\partial t^2} \]  

(12)

In the absence of the distributed axial loads, Eq. (12) turns to be [13]

\[ EA \frac{\partial^2 u(x,t)}{\partial x^2} = \left(1 - \mu^2 \frac{d^2}{dx^2} \right) \rho A \frac{\partial^2 u(x,t)}{\partial t^2} \]  

(13)

Assuming a harmonic vibration for a CNT, the longitudinal displacement may be written as the product of two functions having a single variable by using the separation of variables technique.

\[ u(x,t) = U(x) \sin \omega t \]  

(14)

Plugging assumed solution in Eq. (14) into Eq. (13) gives

\[ \frac{d^2 U(x)}{dx^2} + \frac{\alpha^2}{(1 - \mu^2 \alpha^2)} U(x) = 0 \]  

(15)

where \( \alpha^2 = (\rho A \omega^2)/(EA) \). Most of the references have been dealt with the solution of Eq. (15).

Instead of using the governing equation in the form of Eq. (15), in the present study, the problem is originally to be studied based on the transfer matrix method which is one of the initial value problem (IVP) solution techniques [25-31].

First of all, by assuming \( u(x,t) = u^t(x) \sin \omega t \) and
is called the differential matrix. By introducing the state vector \( \mathbf{x} \), and then the boundary conditions are satisfied. Consequently, the following differential equation.

\[
\frac{d}{dx} \mathbf{F}(x, \omega) = D \mathbf{F}(x, \omega)
\]  

Under the initial conditions

\[
F(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

solution of Eq. (26) gives a closed-form expression of the transfer matrix.

\[
F(x, \omega) = e^{i \Omega x} = \begin{bmatrix} \cos(\frac{x \Omega}{\sqrt{\rho \omega}}) & \sin(\frac{x \Omega}{\sqrt{\rho \omega}}) \\ -\Phi \psi \Omega \sin(\frac{x \Omega}{\sqrt{\rho \omega}}) & \cos(\frac{x \Omega}{\sqrt{\rho \omega}}) \end{bmatrix}
\]

The transfer matrix relates the state vector at any section, \( S(x) \), to the initial state vector at \( x = 0 \), \( S(0) \), as follows.

\[
S(x) = F(x, \omega) S(0), \quad 0 \leq x \leq L
\]

In order to determine the natural frequencies, Eq. (29) is written at \( x = L \), and then the boundary conditions are applied. In the present study two classical boundary conditions namely clamped-clamped (CC) and clamped-free (CF) are to be examined. As is well known, the axial displacement will vanish at at fixed-end while the axial force is zero at a free-end. That is for CC ends

\[
F_{21}(L, \omega) = 0
\]

for CF ends

\[
F_{22}(L, \omega) = 0
\]

should be satisfied. Consequently, the following characteristic frequency equations in the simplest form may be achieved

\[
CF_{\text{local}}(\omega) = \cos\left(\frac{L \omega}{\sqrt{\rho \omega}}\right) = 0
\]

\[
CF_{\text{non-local}}(\omega) = \cos\left(\frac{L \omega}{\sqrt{\rho \omega}}\right) = 0
\]

\[
CC_{\text{local}}(\omega) = \sin\left(\frac{L \omega}{\sqrt{\rho \omega}}\right) = \sin\left(\frac{L \omega}{\sqrt{\rho \omega}}\right) = 0
\]
\[ CC_{\text{non-local}}(\omega) = \sin \left( \frac{L \omega}{\Phi \sqrt{\mathcal{A}}} \right) = \sin \left( \frac{L \omega \sqrt{\rho A}}{\sqrt{EA}} \frac{1 - \mu^2 \rho A \omega^2}{EA} \right) \] (35)

It is worth noting that in the local elasticity theory, \( \Phi = 1 \).

### 3. Validation of the Results

The results obtained in the present study are first compared with the axial natural frequencies of a uniform CC-CNT and CF-CNT rods in Tables 1 and 2.

**Table 1. Comparison of the present natural frequencies of a CC-CNT with the open literature (\( \mu = \mu / L \)).**

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>Kumar and Sujith [32]</th>
<th>Kiani [15]</th>
<th>Li et al. [20]</th>
<th>Xu et al. [22]</th>
<th>Present</th>
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Dimensionless natural frequency in those tables is defined by

\[ \rho^2 = \omega^2 L^2 \frac{P}{E} \] (36)

From Tables 1-2, it is observed that the results are in perfect harmony with each other.

**Table 2. Comparison of the present natural frequencies of a CF-CNT with the open literature (\( \mu = \mu / L \)).**

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>Kumar and Sujith [32]</th>
<th>Kiani [15]</th>
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<th>Xu et al. [22]</th>
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Table 1. Comparison of the determinant-frequency curves of CC and CF rods in both the local and non-local elasticity theories.

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4. Results and Discussion

In this section, a detailed investigation is to be conducted to inspect the differences in the results of both local and non-local elasticity theories and the vibrational parameters on the natural frequencies. The following data is to be used in the analysis (Figure 2).

\[
d = 2r = 1\text{nm} \quad t = 0.34\text{nm} \quad A = \pi d = 0.34\text{nm}^2
\]

\[
\rho = 1000 \text{ kg/m}^3 \quad E = 1\text{TPa}
\]

\[
L = 5\text{nm} ; 10\text{nm}
\]

\[
\mu = 0.25, 0.5, 0.75, ..., 2\text{nm}
\]

Comparison of the determinant-frequency curves of CC and CF rods in both the local and non-local elasticity theories is seen in Figure 3. It is interesting that as the harmonic local longitudinal vibration exhibit an unremitting oscillations under two boundary conditions considered, the non-local harmonic oscillations vanish at a certain frequency value. This cut-off frequency value depends on the boundary conditions and the properties of the CNT-rod. The variation of the cut-off frequency is also illustrated in Figure 4.
Figure 4. Variation of the cut-off frequency in CF-CNT and CC-CNT rods.
Variation of the first seven natural frequencies with the length of CF-CNT and CC-CNT rods in both the local and non-local elasticity theories for $\mu = 2\text{nm}$ is seen in Figure 5. From the figure, in both the theories, as the length of the rod increases, the frequencies in all modes decrease. In the local elasticity theory, there is a linear relationship among the fundamental and higher natural frequencies. However, there is no such a relation between the natural longitudinal frequencies computed by the non-local elasticity theory. The differences between the two successive natural frequencies decrease towards the cut-off frequency in the non-local theory. Consequently, Figure 5 also verifies the existence such a break-frequency in the non-local elasticity theory and it is also compatible with Figures 3 and 4 in this respect.

Variation of the first eight non-local natural frequencies of CF-CNT and CC-CNT rods with the scale factor, $\mu = 0, 0.25, 0.5, 0.75, 2\text{nm}$ for $L = 5\text{nm}$ is shown in Figure 6.

$\mu = 0$ represents the local theory. As seen from Figure 6, when the local elasticity theory is used for natural frequencies of a single-walled carbon nanotube, the highest ones are to be obtained. Taking the scale factor into consideration makes the natural frequencies smaller in the non-local elasticity theory. As the scale factor of a CNT increases, the natural frequencies decrease.

![Figure 5](image_url)  
**Figure 5.** Variation of the first seven natural frequencies with the length of CF-CNT and CC-CNT rods in both the local and non-local elasticity theories.
Figure 6. Variation of the first eight non-local natural frequencies of CF-CNT and CC-CNT rods with the scale factor for L=5nm.
5. Conclusions

In the present study, the longitudinal free vibration of a single-walled carbon nanotube (SWCNT) is formulated and analytically solved based on the Eringen’s non-local elasticity theory and the transfer matrix method. The existence of a cut-off frequency in harmonic vibration of CNTs in the non-local theory is illustrated graphically. The effects of the scale factor on the non-local frequencies are deeply probed in dimensional. A comparison between the local and non-local theories is made. It is observed that the local theory offers highest and unremitting harmonic natural frequencies while they are decreasing with increasing scale factors of a CNT in the non-local theory.

References


