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A Contribution to the Balanced Growth Path Theory of Economic Growth

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Abstract

This paper renews the balanced growth path notion of economic growth originally based on Uzawa's theorem which stipulates that, despite of investment, the balanced growth path, may not exist and must be checked over time. Indeed, the balanced growth path existence is guarantee since the elasticity of substitution between capital and labour equals one, otherwise, it is not and deserves forces for it to exist. The growth literature identifies thus, human capital accumulation to be one of the required forces allowing the balanced growth path existence, thus this article highlights another one, which is automation no more based on the elasticity on substitution level like before when input augmenting technological change is used. So that, in this paper, using the both methods of the balanced growth path emergence evocated, we show several things which are, first when national income factor shares in labor is heterogenous and input augmenting technological change used, second, when machines and unskilled labour share tasks automation are used in an OLG model, where innovations keep rising and capital-skills continuously introduced, labour skills decrease, thus make the unskilled leave the labor market and the engineers switch from production to research. Since, additional power for the BGP existence i.e, ability at work and human capital increase are conducted then, machines' workers exhibit a threshold ensuring long-run growth emergence and social stability.

Keywords

Balanced Growth, Equilibrium, Task Automation, Input Augmenting Technology

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1. Introduction

For a long time, growth theorists looked for the determinants of the long-run growth and established them through the works of Solow [1] for its existence, Denison [2] for its stability through the time and then, Romer [3] and Lucas [4] as well as Aghion and Howitt [5] identify respectively, knowledge, endogenous technological change, human capital and innovations through R&D to cause long-run growth. Moreover, economic performance is proved to be able to reach high levels, depending on human capital investment [6] and can be linked to endogenous technological change [7]. However, factor shares' in national income were left behind despite of their crucial role in the economic growth, in regard to the introduction and the evolution of artificial intelligence, robotics, machines which substitute or complement labour etc called tasks automation in production. The story is revisited by authors such as Gossman and al [8, 9] in case of *input augmenting technological change* i.e something which increases the concerned factor. Whereas, Acemoglu and al [10-14] did the same thing in case of *automation based approach*, leaving emerge the fact that, national labor shares decrease is mainly caused by *tasks automation* in production through artificial intelligence, robotics and all the technologies which substitute or complement labor in production. Those studies established thus, two main methods of studying labor impact on the economic growth i.e *the input augmenting technological change* and *automation of the tasks based-approach,* specifically in order to look for

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the balanced growth path existence and stability in the economy through the time. Moreover, according to the economic literature, labor share in national income decreases, empirically observed from capital-skill complementing labour i.e computer assisted machines, robotics and artificial intelligence that have automatized several parts of the production process in creating several tasks done by machines rather than humans [14, 15]

In order to study those mutations, economic models technological change as factor augmenting or as several tasks to accomplish by machines or by humans, the first vision is used in this article to established the balanced growth path, when machines decrease both skilled and unskilled labor skills, thus the labor demand in good production sector. In that case, technological change evolution in regard to national inputs shares is governed by the elasticity of substitution between capital and labour [16, 8, 9]. In the task-based approach models, automation expands the set of tasks that can be performed by capital leaving emerge three features which are, *first,* the productivity effect, *second,* new tasks increase, *third,* how to deepen more the relationship with capital accumulation where the rental rate increases investment, thus also raise three phases such that, fast automation increase, reduces labor share but increases capital accumulation even when the elasticity of capital and labor is below 1 meaning that, automation doesn't fully explain or cause labour share decline [17] because automation and capital accumulation relationship drives labor share decrease only when the elasticity of substitution between capital and labor is below 1 joining the first view where in that case, equilibrating forces are needed. Workers with heterogenous skills specialized in different automatized tasks, yield inequalities on the wage rate income as well as shortages forthcomings on certain types of skills. Consequently, excessive automation i.e the substitution of machines for tasks previously performed by labour, increases subsidies to capital. Thus, enlarges the gap between the wage rate income and social opportunity cost of labor input. Whereas, in the case of modelling capital-skill complementarity as a capital augmenting technology, if the elasticity of substitution between capital and labor exceeds 1, it decreases labour share [15], otherwise, if the elasticity of substitution is inside the range 0.5 and 1, capital augmenting technological change increases the labour share [18], but, robots have a negative impact on local employment and wages. However, since in automation modelling tasks method, the impact of factors shares in national income doesn't depends on the elasticity of substitution between capital and labor but on capital accumulation which generate different skills and tasks associated with labour productivity power [11], artificial intelligence and robotics boost growth, attaining 0.37% of GDP between 1993 and 2007 for almost 17 countries in average, in the labor concern, automation complements and substitutes labour [19, 20], robots as no effect on labour [21] or automation as an effect on labour [10]. Therefore, the link between capital-skill i.e artificial intelligence, robotics or in a simple word, automation, is tasks expansion [22] and yields workers share at the rate of 47% and automation is identified as partners of machines as well as production function inputs [23]. According to the OECD, automation rise job lost risk at the rate of 9%. The tasks impacted by capital-skill complementarity over the few years hold on new products and process as well as international trade [17]

However, most studies on new innovations in the economic literature, study income factor shares, in considering the labor as a homogenous input of production even dividing work on tasks, there doesn't exist a threshold where some are skilled and others are not, in order to study the difficulties faced by each of workers' category to remain efficient in the labor market. Since, capital-skill complementarity accompanies labor during production, then, *first* it creates unemployment specifically of unskilled labor through a kind of substitution effect, *second,* it increases incentives to invest in human capital accumulation which is costly, thus deserves a financial investment, then raises inequalities in order to achieve the required skill level i.e the one able to keep absorbing and adapting new technology continuously in the production sector rendered discarded after that step and left to the unskilled labor use making some of them leave the labor market when their ability to understand and to work on new technology becomes too low to allow them use those new discarded technological change since their complexity still too high for the non-trained unskilled labor.

Always in conformity with the economic growth literature, stability between factor shares in national income holds, when the elasticity of capital and labor equals 1, meaning that, equilibrating forces are in play. However, Grossman and al [9] proved that endogenous education is the equilibrating force of factors shares in national income when the elasticity of substitution between capital and labor shares is below 1 using *input augmenting technological change methodology*. Otherwise i.e in the case of equilibrating forces absence, once innovations absorbed in the production sector by engineers, then that new technology, once discarded and left to the unskilled labour use [7], the skilled labour demand decreases in the production sector, that they leave for the education sector to do research assuming that only those two sectors are available for them. Therefore, an unbalanced workers' share emerge between the production and the research sectors in income shares evolution conjugated with unemployment of the unskilled labor specifically, then increases investment share expressed by capital or the system automatized more tasks in production, then the equilibrium in factor shares in national income is ruled-out. Indeed, another equilibrium existence and stability is looked for. However, it s proved in [24] that, labour share reaches 47% when automation and labor complement are organized in several tasks to accomplish.

Grossman and al [9], founds of the first part of this article but in contrast, here labor is a heterogenous input, thus the focus of the first part is not only on the skilled labor like in the basic analysis, in contrast, unskilled labor motivations to get and remain in the labor market are also examined in such a way that, the principle of equality between marginal productivity and the wage rate income holds. The question is, in how far can someone considered that, there is a significant proportion of labor in the national income since three factors are considered i.e capital, routine tasks and skilled labor?

The equilibrium looked for, may yields through learning more on new technology while working, not viewed in the literature, specifically in the basic model where incentives to accumulate human capital i.e education only, is considered to be the required force to establish the balanced growth path equilibrium when the elasticity of substitution between capital and labor is below 1. So that, capital-skill complementarity evolution through the time moves such that, at the steady state, crucial variables grow at the constant rate over time.

Indeed, this article extends the basic study in introducing learning-by doing on new technology when working [25, 26] since labour is heterogeneous where, the skilled are assimilated to the engineers in the production sector, who adapt and absorb new technology provided by the research sector generated in the universities by professors and the unskilled perform routine tasks using the technology already absorbed by the engineers that need to be maintain at the threshold level of human capital so that, those least skilled are dynamics in regard to technological change growth. The first equilibrium or the BGP among factor shares looked for, mainly concern heterogenous labor demand instability in the input augmenting technological change method used. The article pends-up its study, with the examination of the impact of the machines on the unskilled labour demand i.e *automation tasks based-approach* where it is shown the existence of both the unique and the multiple equilibria, couples of unskilled' ability on new discarded technology and machines used rate allowing unskilled labor marginal productivity to remain constant through the time. The article is organized like follow, section 2 sets up the basic model, section 3 studies the labour skills' growth rate. Section 4 equilibrate national income factor shares and leaves emerge the balanced growth in order to open-up the determinants of the long-run growth, section 5 presents the automation-based approach where it is shown the threshold existence in machine used allowing the BGP to emerge. Finally, section 6 concludes on the study.

2. The Model

We build a discrete time, overlapping generation model where the agents live during an infinite time i.e both the firms and the consumers where each agent is rational in his choice.

2.1. Firms Optimization

At time, t, the production sector employs physical capital, K_t, labour, $L_t=H_t+U_t$ and capital skill, $(\rho_t^i)_{i=K,H,U}$ to innovate in new goods, using the production function, F such that, $Y_t=F(A_K K_t,\ (u_t h_t)H_t,\ (u_{ts}s_t)U_t)$ where F is continuously differentiable, concave with constant returns to scale, A_K is *capital augmenting technological change* and $B_L=B_H+B_U$ is the sum of the respective *skilled and unskilled labor augmenting technological change* i.e B_H belongs to the skilled labor, and B_U belongs to the unskilled labour such that, $B_H=u_{th}h_t$ whereas, $B_U=u_{ts}s_t$ and u_{th}, u_{ts} ϵ $[0,1]$ are the respective time spent in the production sector by the skilled labor and the unskilled labor. Since, h_t is human capital level, then $L_{th}=B_H H_t$ is skilled labour stock in productivity term, in parallel, since s_t is unskilled labour ability at work such that, $s_t=(1-h_t)\leq h^*$ where, h^* is human capital threshold, then, $L_{ts}=B_U U_t$ is unskilled labour stock in term of productivity in the production sector. Each worker available time is normalized to 1, the rest, $(1-u_{ti})$ is deserved to education for $i>h^*$ and to leisure and sometimes, to learning in new technology for, $i=s_t$ and $h_t\leq h^*$ when the worker is unskilled because his ability at work is, $s_t\leq h^*$. Indeed, the output can be written such that

$$Y_t=F(A_K K_t, B_H H_t,\ B_U U_t) \tag{1}$$

Where, ρ_t^K capital-skill, is the effect of capital accumulation, K_{t+1} on the marginal productivity of labor.

Assumption 1: F is homogenous of degree 1 in K_t as well as in H_t and in U_t and it also exhibits capital-skill, $\rho_t^H=ln(F_H)/ln(K_{t+1})$ i.e the effect of capital accumulation on the marginal productivity of human capital as well as, $\rho_t^U=ln(F_U)/ln(K_{t+1})$ is the effect of capital accumulation on the marginal productivity of routine tasks.

Since innovations are generated by the research sector in universities by the professors first, and capital skill complementarity, arise from intentional investment made by profit maximizing firms on the basis of research discovered by professors, then, the workers' productivity conjugate with capital-skilled complementarity, generate high-tech products in the sector of production where engineers absorb and adapt new innovations in production and once done, technology becomes discarded and left to the use of the unskilled labor in the low-tech sector of production of managing the same homogenous good. The firm production function, (1) is competitive both on capital and labor that are hired up to the point where the respective marginal product equals the rental rate, R_t, for capital and both, w_t^H and w_t^U for the respective skilled and the unskilled wage rate incomes. Defining, $\theta_t^K=R_t(K_t/Y_t)$, as the capital share in national income, then, the respective skilled and unskilled labor shares in national income are, $\theta_t^H=w_t^H(H_t/Y_t)$ for $h_t>h^*$ and $\theta_t^U=w_t^U(U_t/Y_t)$ for $h_t\leq h^*$ where, $\theta_t^K+\theta_t^H+\theta_t^U=1$. Indeed, according to [11], the respective elasticities of substitution between capital and labors are, $\sigma_t^H=dln(K/H)/dln(F_K/F_H)$ for the skilled labour and $\sigma_t^U=dln(K/U)/dln(F_K/F_U)$ for the unskilled labour where F_K, F_H, F_U are the respective derivatives of capital, K, skilled and unskilled labor stocks, H and U. Then, using the definition of $\{\rho_t^K=\rho_t^H+\rho_t^U\}$, $\{\sigma^H,\sigma^U\}$ and $\{\theta_t^K,\theta_t^H,\theta_t^U\}$, as well as optimization of the production function, (1) yield, the rental rates of inputs i.e $R_t=A_KF_K$ for capital $w_t^H=B_HF_H$ for the skilled labour and $w_t^U=B_UF_U$ for the unskilled labor. Indeed, we show that, national income labor shares depend on capital share such that,

$$\theta_t^H=w_t^H(R_t/\theta_t^K)^{1+1/\sigma(H)}(\rho_t^H/Y_tF_K)^{1/\sigma(H)} \text{ and }$$
$$\theta_t^U=w_t^U(R_t/\theta_t^K)^{1+1/\sigma(U)}(\rho_t^U/Y_tF_K)^{1/\sigma(U)}$$

Therefore, labor relative factor shares θ_t^H/θ_t^U expressed by the following equation (2) i.e

$$\theta_t^H/\theta_t^U= (w_t^H/w_t^U)\{(R_t/\theta_t^K)^{1+1/\sigma(H))}/(R_t/\theta_t^K)^{1+1/\sigma(U)}\}[(\rho_t^H)^{1/\sigma(H)}/(\rho_t^U)^{1/\sigma(U)}] [(Y_tF_K)^{1/\sigma(H)}/(Y_tF_K)^{1/\sigma(U)}] \quad (2)$$

Where, $(\rho_t^i)_{i=H,U}$ are machines used to complement labor skills in production and $\sigma(H)=\sigma_t^H$ and the same thing for, $\sigma(U)=\sigma_t^U$. We can see from equations (2) that, labor shares depend on machines, $(\rho_t^i)_{i=H,K}$ through the elasticity of substitution, $(\sigma_t^i)_{i=H,U}$ such that, there is a tendency for labor shares, $(\theta_t^i)_{i=H,U}$ to decrease with national income increase, Y_t and the rental rate of capital, R_t is the winner of the factor shares in national income gain, since it keeps increasing whatever be the rate of the machines used which finally, mostly disturb labor input shares i.e introduce unemployment. Indeed, economic growth measured by the GDP can't be sustained in case of factor shares in income instability i.e the balanced growth path (BGP) doesn't exist.

Proposition 1: *the BGP exist, if and only if, the threshold level in machines used during the production, $(\rho_t^*)_{j=H,U}$ exist, thus is an equilibrating force of national income factor shares expressed by equations, (3) i.e*

$$\rho_t^*=\{(\theta_t^H/\theta_t^U)(w_t^H/w_t^U)\}/(F_H/F_U) \quad (3)$$

Where, $\rho_t^*=\rho_t^{H*}/\rho_t^{U*}$

Proof: since the threshold level in machines used during the production, $(\rho_t^{j*})_{j=H,U}$ is an equilibrating force of national income factor shares expressed by equations, $\rho_t^{H*}=(\theta_t^H/w_t^H)^{\sigma(H)}(\theta_t^K/R_t)^{1+\sigma(H)}Y_tF_K$ and $\rho_t^{U*}=(\theta_t^U/w_t^U)^{\sigma(U)}(\theta_t^K/R_t)^{1+\sigma(U)}Y_tF_U$ yields relative machines used, ρ_t^{H*}/ρ_t^{U*} $=\{(\theta_t^H/w_t^H)^{\sigma(H)}(\theta_t^K/R_t)^{\sigma(H-\sigma(U)}\}/\{(\theta_t^U/w_t^U)^{\sigma(U)}\}(F_H/F_U)$,

Equation (3) means that, the current technological change evolution, doesn't allow production without machines since, $\rho^i=0$, $i=H$, U then, production, Y_t is non-existent. Relative labour shares, θ_t^H/θ_t^U, relative wages and relative marginal productivity are correlated in order to highlight, how capital skilled threshold, ρ_t^* an investment, is managed in order to ensure sustained growth. Thus, innovations through technological change arise, the economy reaches the BGP. Indeed, forces which guaranty the balanced growth are in play through relative wages and relative marginal productivity. Consequently, in case of threshold absence in machine used, then too much capital or machines substitute labor, driving the economy toward a continuous regression, specifically in social term.

Introducing the wage rate income definitions inside equation (3) such that, $\sigma(H)=\sigma_t^H=\sigma_t^U=\sigma(U)=1$, yields, $\rho_t^{H*}/\rho_t^{U*}=\{(\theta_t^H/\theta_t^U)(w_t^H/w_t^U)\}/(F_H/F_U)=(B_H/B_U)(\theta_t^H/\theta_t^U)$ and introducing now equation (2) inside, yields, equation (4) i.e

$$F_H/F_U=(B_H/B_U)^{-2} \quad (4)$$

Equation (4) relates the fact that, relative marginal productivity depends on relative input augmenting technological change in such a way that, too much technology or machines in production, decreases relative labour marginal productivity. The same thing holds if too much skilled increase i.e high technology increase, B_H decrease relative marginal productivity which thus, decrease more the unskilled capability at work.

2.2. Input Augmenting Technological Change Explanation

Skilled labor augmenting technological change impact on the equilibrium wage, $dln(w_t^H)/dln(B_H)=w_t^HH_t/\sigma_t^HY_t=1-dln(w_t^H)/\sigma_t^H>0$ since, $\sigma_t^H>dln(w_t^H)$ then, B_H increases the skilled labor equilibrium wages when the elasticity is high. In parallel, unskilled labor augmenting technological change impact on the equilibrium wage, $dln(w_t^U)/dln(B_U)=w_t^UU_t/\sigma_t^UY_t=1-dln(w_t^U)/\sigma_t^U>0$ if, $\sigma_t^U>dln(w_t^U)$ then, B_U increases the unskilled labor equilibrium wages when the elasticity is high. Capital augmenting technological change impact on the skilled and the unskilled labors respective wages rate incomes,

$dln(w_t^H)/dln(A_K)=(w_t^H H_t/Y_t\sigma_t^K)=1-dln(R_t)/\sigma_t^K>0$ and $dln(w_t^U)/dln(A_K)=(w_t^U U_t/Y_t\sigma_t^K)=1-dln(w_t^H)/\sigma_t^K>0$ can be viewed such that, $\sigma_t^K>dln(w_t^H)$ and $\sigma_t^K>dln(w_t^U)$ then, A_K increases both the skilled and the unskilled labour shares i.e their income and their respective labor demand if and only if $\sigma_t^H>0$ and $\sigma_t^U>0$. Indeed, according to Oberfield and Raval [18], constant returns make K on the one hand and (H, U) on the other hand, be complements such that, the impact of A_K on the labour shares, $w_t^H H_t/Y_t$ and $w_t^U U_t/Y_t$ be expressed by,

$$dln(w_t^H H_t/Y_t)/dln(A_K)=(w_t^H H_t/Y_t)(1/\sigma_t^H-1)<0 \text{ if } \sigma_t^H>1$$

$$dln(w_t^U U_t/Y_t)/dln(A_K)=(w_t^U U_t/Y_t)(1/\sigma_t^U-1)<0 \text{ if } \sigma_t^U>1$$

Thus yield, capital augmenting technology, A_K decreases the labor shares if the elasticity of substitution is greater than 1 [16]. Otherwise, if the elasticity of substitution is such that, $0.5<\sigma_t^H<1$ and $0.5<\sigma_t^U<1$ then A_K increases labor shares since the elasticity of substitution between labor and capital is less than 1 [19]. Indeed, B_H and B_U increase labor demand as well as the associated income.

2.3. Utility Optimization

The agents' utility function is expressed such that, $U_t=\sum_{t=0}^{+\infty}(\beta_t^i c_t^i+\alpha_t^i q_t^i)$ where, $0<\beta_t^i$, $\alpha_t^i<1$ are the elasticities of investment on per-capita financial assets, q_t^i and on per-capita consumption c_t^i of the respective skilled, $i=H$ and unskilled, $i=U$. The agents earn the wage rate income, w_t^i for $i=H, U$ that they spend on consumption and on financial assets which bring dividends, d_{t+1}^i next period. Indeed, the intertemporal budget constraint at each time, t is expressed such that, $q_t^i+c_t^i=d_{t+1}^i+w_t^i$ for $t\epsilon[0,+\infty[$, $i=H, U$.

Then, the representative agent program is expressed such that,

$$Max\{U_t=\sum_{t=0}^{+\infty}(\beta_t^i c_t^i+\alpha_t^i h_t^i)\}$$

$$sc$$

$$q_t^i+c_t^i=d_{t+1}^i+w_t^i(h_t^i)$$

$$t>0$$

The first order conditions of utility optimization provide the respective equilibrium in per-capita consumption and financial assets,

$$c_t^{i}*(w_t^i)=\beta_t^i/(\alpha_t^i+\beta_t^i)[w_t^i+d_{t+1}^i] \text{ and } q_t^{i}*(w_t^i)=\alpha_t^i/(\alpha_t^i+\beta_t^i)[w_t^i+d_{t+1}^i], \text{ where, } i=H, U$$

The respective skilled and unskilled agent chooses time allocation, u_{tH}, u_{tU} to maximize the expected present value of earnings where both human capital and unskilled labor accumulation move following [9], a function such that, for an individual born at t, the problem is

$$Max\{\sum_t^{+\infty}(u_{ti}h_{ti}(w_t^i)+d_{t+1}^i q_t^i(w_t^i))\}_{i=H,U}$$

$$sc$$

$$h_{t+1}-h_t=(r_t+v_{tH})h_t+u_{tH}q_t^H(w_t^H)$$

$$s_{t+1}-s_t=(r_t+v_{tU})s_t+u_{tU}q_t^U(w_t^U)$$

The first order conditions of the previous problem yield,

$$h_{t+1}-h_t=(r_t+v_{tH})h_t+(d_{t+1}^H(r_t+v_{tH})^{1/2}q_t^H(w_t^H)$$

$$s_{t+1}-s_t=(r_t+v_{tU})s_t+(d_{t+1}^U(r_t+v_{tU})^{1/2}q_t^U(w_t^U)$$

Where r_t is the real interest rate and v_{tH} and v_{tU} are the instantaneous risks of the respective skilled and unskilled labor skill decrease caused by technological change evolution in the production sector.

3. The Labour Skills Growth Rates

Lemma 1: according to the first order conditions of the previous agent program, the respective skilled and unskilled skills' growth rates, g^H and g^U are given by the following equations (5) and (6) i.e

$$g^H=(r_t+v_{tH})+d_{t+1}^H(r_t+v_{tH})^{1/2}q_t^H(w_t^H/h_t) \quad (5)$$

$$g^U=(r_t+v_{tU})+d_{t+1}^U(r_t+v_{tU})^{1/2}q_t^U(w_t^U/s_t) \quad (6)$$

Definition 1: the average skills economic growth rate, g is expressed such that, $g=1/2(g^H+g^U)$

Proposition 2: *according to definition 1 the average labour skills growth rate of the economy, $g=1/2(g^H+g^U)$ is expressed by the following equation (7) i.e*

$$g=(\pi_t/2)(2+d_{t+1}\pi_t^{-1/2}W_t) \quad (7)$$

Proof:
$g^H+g^U=g=(r_t+v_{tH})+(d_{t+1}^H(r_t+v_{tH})^{1/2}q_t^H(w_t^H/h_t)+(r_t+v_{tU})+(d_{t+1}^U(r_t+v_{tU})^{1/2}q_t^U(w_t^U/s_t)$

Assuming that dividends provided by financial assets are the same through the time in average since financial assets are standardized then, $d_{t+1}^H=d_{t+1}^U=d_{t+1}$ and the skills lost risks are also the same through the time i.e $v_{tH}=v_{tU}=v_t$

then, the previous equation becomes, $g=(r_t+v_t)+d_{t+1}(r_t+v_t)^{1/2}(q_t^H(w_t^H/h_t)+q_t^U(w_t^U/s_t))$

$=(r_t+v_t)(2+d_{t+1}(r_t+v_t)^{-1/2}(q_t^H(w_t^H/h_t)+q_t^U(w_t^U/s_t)/2)$

Setting, $r_t+v_t=\pi_t$ the labour force skills' growth rate growth rate becomes,

$$g=(\pi_t/2)(2+d_{t+1}\pi_t^{-1/2}(q_t^H(w_t^H/h_t)+q_t^U(w_t^U/s_t))$$

$$=(\pi_t/2)(2+d_{t+1}\pi_t^{-1/2}W_t)$$

Where, $W_t=q_t^H(w_t^H/h_t)+q_t^H(w_t^H/s_t)$ is total wealth in skills efficiency.

Proposition 2 links skills growth rate to wealth in efficiency units meaning that, labor is associated to a lost risk of employment supplemented by the interest rate. Capital skill complementarity is absent in labor productivity increase which depends on human capital, h_t as well as simple work, s_t in contrast which are correlated with investment. Wealth measured by the product of the wage rate income per efficiency and financial assets hold increases the average skills rate since they allow for more investment on human capital accumulation and on ability to understand new technological change through education, learning while working or training. Thus, technological progress speed highlights by capital skills complementarity yields perpetual investment on skills hidden in the average skills growth rate. Once coupled with capital or investment which level is costly, then the economy is boosted, thus decreases labor shares in national income or yields inequality. In case of instability of national income shares or in case of skills insufficiency, the equilibrium between education and production sectors rules-out caused by the unskilled labor unemployment constant increase through the time with the skilled labor perpetual human capital accumulation explaining productivity in hours increase compare to the past centuries [27]

Assumption 2: capital stock and labor skills depreciate through the time at the rate, $(\delta_i)_{i=K,H,U}>0$ *i.e* $K_{t+1}-K_t=I_t-\delta_i K_t$, $H_{t+1}-H_t=I_H\rho_t^H-\delta_H H_t$ *and* $U_{t+1}-U_t=I_U\rho_t^U-\delta_U U_t$ *where* I, I_H *and* I_U *are the respective investment in capital, human capital accumulation and ability increase while working being trained as well as learning*[1]

Lemma 2: suppose that, g is constant, then, along any BGP with $0<C_t<Y_t=C_t+I_t$, $n>0$ *where new installed machines (units of capital skilled),* $I_t=n\rho_t>0$, *good consumption stock,* $C_t=nc_t>0$ *and* $0<H_t<U_t$, *then the equilibrium such that,* $g=1/2(g^H+g^U)=g_c=g_K=g_Y$ *exist*

Proof: see [8]

Lemma 2 states that, the path where the growth rates of consumption, capital, output and heterogenous labors grow at the same constant rate such that, the value of capital is measured at the unit of consumption goods, exist. Similarly, lemma 2 provides the existence of the BGP.

Lemma 3: if total-augmenting technological progress, $\gamma_K=A_K+B_H+B_U$ *and the elasticity of substitutions,* σ_t^H *and* σ_t^U *are such that, both education and training and/or learning are fixed, since, the BGP exist (lemma 2), then the input factors are remunerated at their marginal productivity i.e* $w_t^{H*}=(1-\sigma_t^H)\gamma_K=\sigma_t^H(F_H/F_K)\partial(F_H/F_K)/\partial(K/H)h_0$ and

$$w_t^{U*}=(1-\sigma_t^U)\gamma_K=\sigma_t^U(F_U/F_K)\partial(F_U/F_K)/\partial(K/U)s_0$$

Lemma 3 stipulates that, the BGP with constant factor shares might exist despite of the fact that, the elasticities of substitutions are not equal to 1, thus $h_0\neq 0$ and $\partial(F_H/F_K)/\partial(K/H)\neq 0$ (respectively, $s_0\neq 0$ and $\partial(F_U/F_K)/\partial(K/U)\neq 0$). Lemma 3 also highlights the interaction between capital and labor augmenting technological change on the one hand and the change both in education (first expression), as well as in learning by-doing or in training (second expression) on the other hand. If h_0 human capital index is constant, i.e $h_{t+1}=h_t$ (respectively, s_0 is constant i.e $s_{t+1}=s_t$) then, a BGP with constant and strictly positive factors can exist only if, $\sigma_t^H=1$ or $\gamma_K=0$ (respectively, $\sigma_t^U=1$ or $\gamma_K=0$) since in the BGP of a neoclassical growth model without population growth and exogenous technological progress is simply, a Cobb-Douglas production function. When both indexes of human capital and ability at work are included in the neoclassical growth model with input augmenting technological progress, then, there exist a measure of both human capital and ability at work, such that, the BGP with constant and strictly positive factor shares exist when $\sigma_t^H=1$ or $\gamma_K=0$ (respectively, $\sigma_t^U=1$ or $\gamma_K=0$). If the elasticities of substitution are below 1, then, the BGP existence, requires, $\partial(F_H/F_K)/\partial(K/H)h_0>0$ (respectively, $\partial(F_U/F_K)/\partial(K/U)s_0>0$) i.e an increase in capital stock, must increase the marginal product of education (respectively, increase the marginal product of ability at work) for the technology to be capital-skilled complementarity.

Corollary 1: since $k_1=H_t/K_t$ *and* $k_2=U_t/K_t$ *are constant, then,* k_1/k_2 *is also constant, indeed, a BGP with constant and strictly positive factor shares can exist, only if* $\sigma_t^H/\sigma_t^U=1$ *i.e* $\sigma_t^H=\sigma_t^U=1$ *and* $\gamma_K=0$ (see [29] for proof)

Proposition 3: *in case of heterogenous labor in an input augmenting technological change, there exists an aggregate elasticity of substitution,* σ *expressed by equation (8) i.e*

$$\sigma=(B_H/B_U)/(h_0/s_0) \quad (8)$$

Proof: according to lemma 3, relative wages are expressed such that, $w=\sigma(F_H/F_U)(h_0/s_0)$, since, $F_H/F_U=(B_H/B_U)^{-2,}$ $\sigma=\partial(F_H/F_U)/\partial(U/H)$ and $(\sigma_t^H/\sigma_t^U)=1$ then, $w=w_t^H/w_t^U=B_HF_H/B_UF_U=(B_H/B_U)=\sigma(h_0/s_0)$ indeed, $\sigma=(B_H/B_U)/(h_0/s_0)$

Equation (8) highlights forces that play for national income and the associated shares to remain in balanced i.e both human capital and ability at work indexes as well as, labor augmenting technological change.

To close the model, we set the average growth rate of the labor skills at time at t, equation (7) to be equal to relative wages, provided by lemma 3 such that, $g=w_t^{H*}/w_t^{U*}$ where i.e $(F_H/F_U)(B_H/B_U)=g=(\pi_t/2)(2+d_{t+1}\pi_t^{-1/2}W_t)$ and $\sigma=(B_H/B_U)(h_0/s_0)$ which yields, the closure of the model

1 $g_K=(K_{t+1}-K_t)/K_t=I_t/K_t-\delta_i$, $g^H=(H_{t+1}-H_t)/H_t=I_H\rho_t^H/H_t-\delta_H$ and $g^U=(U_{t+1}-U_t)/U_t=I_U\rho_t^U/U_t-\delta_U$

equation through wealth expression provided above such that equation, (9) i.e

$$W_t=(2\sigma F_H/(h_0/s_0)\pi_t F_U-2)/d_{t+1}\pi_t^{-1/2} \quad (9)$$

Where, $\phi=(F_H/F_U)$ is relative marginal productivity ratio of the skilled and the unskilled. Equation (9) closes the model and highlights the fact that, wealth depends on the aggregate elasticity of substitution, σ and relative human capital and ability at work indexes ratio, (h_0/s_0) such that, the both variables, increase wealth according to some restrictions. Since, we evidently, have $2\sigma\phi(h_0/s_0)/\pi_t\geq 2$ i.e $\sigma\geq(\pi_t/\phi)/(h_0/s_0)$ for the theory to hold. Indeed, we can discuss the BGP existence and stability depending on the aggregate elasticity of substitution, σ such that, if $\sigma=1$ then, $W_t=[2\phi(h_0/s_0)/\pi_t)-2]/d_{t+1}\pi_t^{-1/2}$ therefore, h and s grow at the same constant rate, g^0 and then, a BGP with constant and strictly positive factor shares making income, W_t increase at a constant rate can exist. Otherwise, if, $\sigma<1$, i.e there exist $\varepsilon_1>0$ such that, $\sigma=1-\varepsilon_1<1$ then, $W_t=[2(1-\varepsilon_1)\phi(h_0/s_0)/\pi_t)-2]/d_{t+1}\pi_t^{-1/2}$ therefore, equilibrating forces are needed for the BGP to hold, since income factor shares, W_t grows at the constant rate which can be education or ability at work as well as learning more accomplishments whenever, $\sigma>0$. Finally, if $\sigma>1$ then, $W_t=(2(1+\varepsilon_2)\phi(h_0/s_0)/\pi_t)-2)/d_{t+1}\pi_t^{-1/2}$ where, $\varepsilon_2>0$ then, labor augmenting technological change increases the equilibrium income, W^* since the aggregate elasticity is high.

4. The BGP in National Income Factor Shares

The equilibrium existence among factors' share in national income, means to generate a balanced growth path which to emerge, needs taking care of several things, *first,* since, capital-augmenting labor skills through technical progress doesn't hold with constant returns in a standard neoclassical growth model [28, 29] solved by [30] who showed that it yields, the equilibrium with monopolistic competition. *Second,* a decreasing rate of return on capital, decreases skills i.e both human capital level and simple work's productivity, then, incentives to increase skills rise i.e education and training while working for example in order to maintain a stable economic growth factors' shares in national income. *Third,* education or human capital accumulation increase, rises simple labor force skill decrease because of capital skill complementarity introduction in production, then maintaining a constant factor shares in national income needs forces able to ensure it i.e a BGP existence must be looked for i.e we need to find a path, where capital both physical and embodied knowledge added to simple tasks, accumulate or move at the same constant rate. Since the model is closed, we can now, generate a BGP using the last equation.

To generate the long-run growth path, we need to introduce input augmenting technological change again into the model, $Y_t=F(A_K K_t, B_H H_t, B_U U_t; h, s)$ units of output at time, t where A_K is disembodied knowledge i.e capital augmenting technology, B_H is the skilled embodied knowledge i.e skilled labour augmenting technology, whereas, B_U is the unskilled embodied knowledge i.e unskilled labour augmenting technology where labour is heterogeneous i.e, $L_t=h_tL_t+s_tL_t=H_t+U_t$. Indeed, in order to generate a BGP in a neoclassical model, we begin by announcing assumption 3

Assumption 3: according to [8], the production function exhibits constant returns and can be written such that,

$F(A_K K_t,(B_H/B_U)(H_t/U_t))$

$=G(e^{-a(h/s)}A_K K_t, e^{b(H)h/b(U)s}(B_H/B_U)(H_t/U_t))$ such that, the following assertions are satisfied

(i) *$f(k)\equiv G(k, 1)$ is strictly increasing, twice differentiable and strictly concave for all, $k=k_1/k_2$ where $k_1=H_t/K_t$ and $k_2=U_t/K_t$*

(ii) *$\lim_{k\to 0} kf'(k)/f(k)<b/a+b$ where $b=b_H/b_U$*

Assertion (*ii*) means that, an increase in relative skills i.e the ratio of human capital and of simple work, increase the demand for capital relative to that of labor at the initial factor prices. Moreover, human capital decreases the productivity of physical capital and capital-skill complementarity drives simple work toward routine task, thus make them leave the labor market quicker replaced by technological change which needs more skilled labor. Assumption 3 is similar to re-writing the production function such that,

$$F(A_tK_t, (B_H/B_U)(H_t/U_t))$$
$$=((B_H/B_U)(H_t/U_t))^{1-\beta}P(A_tK_t, e^{(b(H)h/b(U)s)/\beta}Q((B_H/B_U)(H_t/U_t))^{\beta}$$

Where, $\beta=b/a+b>0$

The previous production function expresses output as a Cobb-Douglas function of unskilled and skilled labors connected to physical capital measuring the workers skills, then marginal productivity of physical capital is raised by human capital level, therefore, the marginal productivity of human capital is positive for all K_t, H_t but slowly decreasing for U_t

As in [31], since a unit of output produces a unit of consumption good, c_t or the investment, q_t then the output is, $Y_t=C_t+I_t/q_t$ where capital accumulation is, $K_{t+1}-K_t=I_t-\delta_K K_t$, C_t and K_t are the respective aggregate consumption and aggregate capital stock, I_t is the gross investment, $\delta_K>0$ is a constant capital depreciation rate.

Let relative labour technological change be such that, $\gamma_H/\gamma_U=((B_H^{t+1}-B_H^t)/B_H^t)/((B_U^{t+1}-B_U^t)/B_U^t)$ i.e the respective constant rates evolutions of the skilled and the unskilled labor-augmenting technological progress where technological progress, $g_A=(A_{t+1}-A_t)/A_t$ is the constant rate of disembodied capital-augmenting progress, $g_q^H=(q_{t+1}^H-q_t^H)/q_t^H$ and $g_q^U=(q_{t+1}^U-q_t^U)/q_t^U$ are the respective constant rates of the skilled and the unskilled investment growth rates. Then, capital augmenting technological progress evolution rate is such that, $\gamma_K=g_A+g_q^H/g_q^U$ the total rate of augmenting technological progress in growth rate of the nation's capital (physical and financial) between parameters that describe the growth process and the long-run growth exists.

Characterizing the BGP

Some assumptions on the parameters are provided, in order to put some restrictions

Assumption 4: the parameters of the economy satisfy (i) [2] *, (ii)* [3] *and (iii)*[4] *defined such that*[5]*,*

(i) $a>\gamma_K$

(ii) $\lim_{k\to 0} kf(k)/f(k)>\Omega/1+\Omega>\lim_{k\to\infty} kf((k)/f(k)$

Where, $\Omega=M(H)/M(U)$

(iii) $(\eta-1)m(\Omega)>0,$

Where, $\eta-1=(\eta_H-1)/(\eta_U-1)$

Assumption 4 ensures the existence of the equilibrium and also raises an interior choice solution for incentives to accumulate human capital as well as optimal simple work training to follow new innovations in the production function.

The competitive firm considers, the rental rate as an exogenous component and hires labour i.e human capital, H_t as well as routine tasks, U_t in combining them with units of relative physical capital, $\varkappa_t(h)$ and $\varkappa_t(s)$, of the skilled and the unskilled expressed such that,

$$R_t=e^{-a(h/s)}A_tG_K(e^{-a(h/s)}A_t\varkappa_t(h/s),\ e^{b(h/s)}(B_H/B_U))$$

Where, $\varkappa_t(h/s)$ is units of physical capital combine with relative labor at time, t

Then, relative wage rate income, $w=w_t(h/s)$ equals the marginal productivity i.e the difference between revenue and capital cost i.e

$$w_t(h/s)=G(.)-e^{-a(h/s)}A_t\varkappa_t(h/s)G_K(.) \quad (10)$$

Thus, relative wage rate income is the response of the individuals' investment in education to become skilled or not to be depending on the level achieved.

Definition 2: a BGP is a dynamic equilibrium where the following variables, constant growth rate of output, consumption, capital (financial and physical), labor inputs grow at a constant and strictly positive rate along the steady state i.e $Y_{t+1}/Y_t=c_{t+1}/c_t=K_{t+1}/K_t=H_{t+1}/H_t=q_{t+1}/q_t=g^*$

Lemma 4: according to definition 2 and assumptions 2, 3, 4 if g_q^H/g_q^U *and* γ_H/γ_U are constants along the steady state, then, there exist a unique BGP characterized by equations (11), (12) i.e,

$$h_t^*/s_t^*=1-\gamma_K/a \quad (11)$$

$$z_t\equiv e^{-a(h/s)}A_tK_t/e^{b(h/s)}(B_H/B_U)(H_t/U_t)=z^* \text{ for all t} \quad (12)$$

Here[6] z_t means that, both human capital and simple work productivity increase with time, it also adjusts the respective capital-labor ratio of the skilled and the unskilled labour i.e U_t/H_t and A_tK_t/B_tL_t for the existing skilled and unskilled labor when different complementary between labor and embodied knowledge are taken account. Finally, the equations, (11) and (12) mean that, both human capital and routine labor thresholds increase linearly with capital evolution rate, γ_K. Since, $h_t^*=\gamma_K/a$ then, optimal human capital when reaches h^* meaning that, the unskilled enters in the production sector since $h^*\leq s_t$ in contrast, when human capital level is such that, $h^*<h_t$ then students deserve a fraction, of more time for human capital accumulation until the cross of the basic level, h^* and continue education. Therefore the skills accumulates such that, $h_{t+1}^*=h_t^*+(\gamma_K/a)(1-u_{th})$ thus human capita level achieved increases linearly such that, $h_{t+1}^*=\gamma_K/a-\gamma_K$ meaning that, human capital level achieved, increases when $a>\gamma_K$ and reaches the steady state, if and only if capital-augmenting technical progress is strictly positive.

Let, $(1-u_{ts})$ be the training time in new technology by the unskilled labor, then if it catch-up human capital threshold, h^* thus simple work also increases linearly such that $s_t=\gamma_K/a-\gamma_K$ where simple work accumulates such that, $s_{t+1}^*=s_t^*+(\gamma_K/a-\gamma_K)u_{tU}$

The second serial equations of lemma 2 i.e (10) and (11) mean that, human capital accumulation and labor ratio converge toward a constant value, z_t^{H*} whereas, routine task converge toward the constant value, z_t^{U*} explaining the BGP existence when the production function exhibits capital-augmenting technological progress with an elasticity of substitution between capital and labor less than 1. Because, capital accumulation along time, increases its productivity

2 *Since,* $\lim_{k1\to 0}\ k_1f'(k_1)/f(k_1)>\Omega_H/1+\Omega_H$ *and* $\lim_{k\to 0}k_2f'(k_2)/f(k_2)>\Omega_U/1+\Omega_U$ indeed,
$\lim_{k(1)/k(2)\to 0} kf(k)/f(k)>\Omega/1+\Omega$

3 *Where* $k_1=K/H$ *and* $k_2=K/U$, $(k_1/k_2)(f'(k_1))/f'(k_2))/f(k_1)f(k_2))=kf'(k)/f(k)>\Omega/1+\Omega$
$\Omega=\Omega_H/\Omega_U$, $k=k_1/k_2$, $f'(k)=f'(k_1)/f'(k_2)$, $f(k)=f(k_1)/f(k_2)$
$\Omega_H=(b_H-\lambda_H/a)-(\eta_H-1)(\gamma_H+(b_H-\lambda_H)\gamma_k)+(\rho^H-(\lambda_H-v_H))/a-\gamma_K$
$\Omega_U=(b_U-\lambda_U/a)-(\eta_U-1)(\gamma_U+(b_U-\lambda_U)\gamma_k)+\rho^U-(\lambda_U-v_U))/a-\gamma_K$

4 *(iii)* $(\eta-1)m(H)$, *where,* $m(H)[(\gamma_H+(b-\lambda_H)\gamma_K/a)+\rho^H-(\lambda_H-v_H))$, $M(U)=(\gamma_U+(b-\lambda_U)\gamma_K/a)+\rho^U-(\lambda_U-v_U))]$

5 *(iii)* $(\eta-1)m(H)$, *where,* $m(H)[(\gamma_H+(b-\lambda_H)\gamma_K/a)+\rho^H-(\lambda_H-v_H))$, $M(U)=(\gamma_U+(b-\lambda_U)\gamma_K/a)+\rho^U-(\lambda_U-v_U))]$

6 $z_t^H=e^{-ah}A_tK_t/e^{bh}B_U^tH_t$
$z_t^U=e^{-as}A_tK_t/e^{cs}B_U^tU_t$

and capital share in national income tends to decrease when the elasticity of substitution is less than 1. However, capital skill complementarity means that, human capital productivity decreases but routine tasks skills decrease more, which put pressure on the factors share and become unbalanced, thus more education is needed in order to maintain as high skill labor as possible in the labor market able to follow new technology when entering in the production sector. Then, it is crucial, for workers of the two kinds to increase their skill through more human capital or a deeply training in new technology in order to keep $(z_t^j)_{j=H,U}$ constant (see figure 1 for illustration).

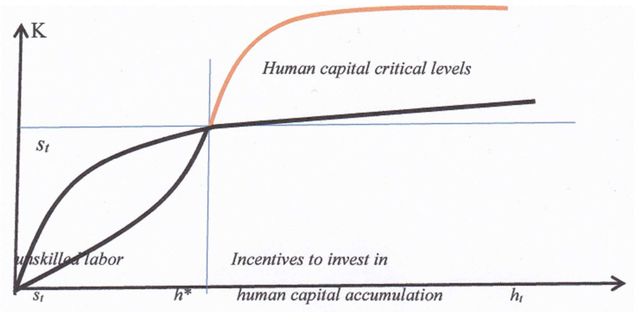

Figure 1. Labour shares evolution.

Moreover, along the BGP, the respective skilled and unskilled wage rate income growth increases at a the following rates, $g_w = g_{w(h/s)} = \gamma_H/\gamma_U + [(b(h/s))/a]\gamma_K$

Since along the BGP, factor shares are constants, then, the output growth rate is expressed such that, $g_y = g_{w(h/s)} = \gamma_H/\gamma_U + [(b(h)+b(s)-\lambda)/a]\gamma_K$ where, $\lambda>0$ is the probability for a living agent to generate a descendent. The proportionality of per-capita consumption to per-capita output, yields the long-run interest rate, $r=\rho+\eta g_y=\rho+\eta(\gamma_H/\gamma_U+[(b(h/s)-\lambda)/a]\gamma_K)$ where $\eta>0$ is the inverse of the elasticity of intertemporal substitution. Finally, in the steady state, the equation which closes the model yields, $\gamma_H/\gamma_U+[(b(h/s))/a]\gamma_K=r+\upsilon-(1-\gamma_K/a)(b(h/s)-a(\theta/1-\theta))$

The long-run capital share and the parameters of the economy is expressed such that

$$\theta/1-\theta=(b(h/s)-\lambda)/a)-(\eta-1)[\gamma_H/\gamma_U+(b(h/s))-\lambda)/a]\gamma_K-\lambda+\upsilon+\rho)/a-\gamma_K \quad (13)$$

Consequently, output growth if at the steady state, is relied to income factor shares parameters. Where, $\theta=\theta_t^K$ is capital share in national income, ρ is the subjective discount rate and υ is the average job lost risk, such that, since in the economic literature, per-capita consumption growth rate is, $g_c=(r_t-\rho)/\eta$.

Summary of the characterization of the BGP

Let the aggregate production function satisfies assumptions, such that, g_A and γ_H/γ_U are constants, then there exists a unique BGP along with all the young are full time students until their human capital level reaches the threshold, h^* that grows linearly with time. After, some of them work for a fraction, $1-\gamma_K/a$ while the others keep staying in the education sector to accumulate human capital which increase more than the threshold. The respective skilled and routine tasks wage rate income grows at the constant rates, $\gamma_H/\gamma_U+(b(h/s)/a)\gamma_K$ and per-capita income grows at constant rate, $\gamma_H/\gamma_U+(b(h/s)-\lambda))\gamma_K/a$

5. The Automation Based on Task Approach

$Y^U=F(,)$ aim is to explain how, the unskilled labor compete with automation, ρ^U in the low-tech production sector such that, the production function highlights a combination of services inside a range of continuum tasks such that,

$$Y^U=(\int_{i=1}^{N} y(i)^{\sigma-1/\sigma}di)^{\sigma/\sigma-1} \quad (14)$$

Where, σ^* is the elasticity of substitution between tasks, 1 to N and automation is denoted, ρ^U such that, if task, $i>\rho^U$ i.e the task is accomplished by labor and not machines, then it yields, $y^U(i)=\gamma(i)l(i)$ where y is per-capita production, γ is labor productivity and l is labor per-efficiency unit. If $i\leq\rho^U$ then, the task to accomplish is done by machines or by labor, then per-capita production function is now expressed such that,

$$y^U(i)=\rho^U(i)k(i)+\gamma(i)l(i) \quad (15)$$

Where, ρ^U are machines productivity making unskilled labor per-efficiency unit, l as perfect substitute inputs in production. Excessive automation production function can be written such that, $Y^U=(\int_{1}^{N}\rho^U(i)^{\sigma-1/\sigma}di)^{\sigma/\sigma-1}$

Since automation or machines, ρ^U expansion means labour substitution, their choice, depends on the comparative cost generated by their use rather than labor in production, in order to study that, we assume that, tasks $[1,I]$ are accomplished by machines, ρ^U and tasks, $]I,N]$ are accomplished by unskilled labor such that, the production function, be the sum of the both workers i.e

$$Y^U=(\int_{i=1}^{I}\rho^U(i)^{\sigma-1/\sigma}di)^{1/\sigma}K^{\sigma-1/\sigma}+(\int_{i=I}^{N}\gamma(i)^{\sigma-1}di)U^{\sigma-1/\sigma} \quad (16)$$

Since, machines, ρ^U impact on the low-tech production sector is expressed by,

$$dln(Y^U)/\rho^U=(1/1-\sigma)[(w^U/\gamma(i))^{1-\sigma}-(R/\eta(i))^{1-\sigma}]>0$$

Where the share of unskilled labor in the low-tech production function is,

$$\theta^U=[1+(\int_{i=N-1}^{I}\rho^U(i)^{\sigma-1/\sigma}di)^{1/\sigma}K^{\sigma-1/\sigma}/(\int_{i=I}^{N}\gamma(i)^{\sigma-1}di)U^{\sigma-1/\sigma}]^{-1}$$

Therefore, unskilled labour share decreases in ρ^U i.e the machines at work but increases in U depending on the level of γ, thus no more on the elasticity of substitution as stipulated by the previous study through *factor augmenting technological change*.

Moreover, assuming, $N=\sigma=1,$ then the low-tech sector production function, Y^U becomes,

$$Y^U=D(K/\rho^U)^{\alpha}(U/1-\rho^U)^{\beta} \quad (17)$$

Where, $\alpha+\beta=1$ and $0<\alpha, \beta<1$

If $\gamma(I)/\rho^U(I)>w^U/R$ then, tasks near I will be substitute to labor and produced by machines. Otherwise, if $\gamma(I)/\rho^U(I)\leq w^U/R$ then, tasks far from I will be substitute to machines and produced by unskilled labor. Since the both workers are perfect substitutes, growth may increase whatever be the choice done but excess automation decreases labor share in national income, Y measured by the GDP. Therefore, there is growth coupled with unemployment if the second hypothesis is retained rather than the first. That fact may rise social disorders or increase government social aids to poor people whom skills no more match with current technological change.

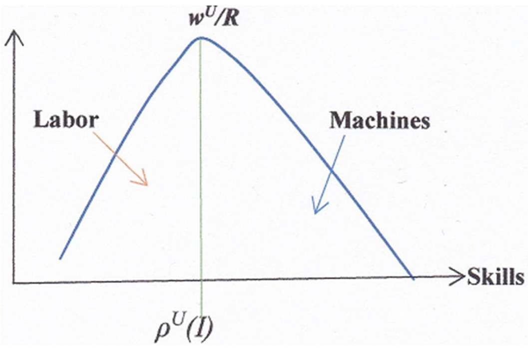

Figure 2. Automation versus Labour.

Since equation (17) isolates automation and unskilled, we announce the last result that shows how the both variables are linked and how to make them grow the same.

Proposition 4: *according to equation (17), since machines used and unskilled labor skill move in the opposite way, then, if the movements are monotonous, then there exist a unique equilibrium,* (u^*,ρ^{U*})*. Otherwise, if their movements are stochastics, then there exist multiple equilibria in machines used and labour demand,* $(u_{tk}, \rho_{tk}^U)_{k\in N}$

Proof: dividing (17) by capital stock i.e computing, $Y/K,$ and setting the derivative of the log of unskilled stock per-capita efficiency to zero i.e doing, $\ln(\partial(Y/K)/\partial(U/K))=0,$ then it yields per-capita efficient unskilled labour ability, u^* i.e,

$$u^* =(\beta\xi(\rho^{U^*})D)^{1/1-\beta} \quad (18)$$

Where, $\xi(\rho^U)=((\rho^U)^{\alpha}(1-\rho^{U^*})^{\beta})^{-1},$ $D>0,$ $u=U/K$ and $0<\rho^U<1,$ indeed, since automation, ρ increase, yields u decrease, therefore, since the one is monotone increasing whereas the other is monotone decreasing, then it can be found, $\rho^U{*}$ that meets, u^* somewhere on the space time, once such that, the both variables grow at the same constant rate over time. Proposition 4 first part, refers to figure 3 where ceases national factors share instability mostly caused by the unskilled labour skills' decreasing i.e unskilled labour marginal productivity decreases over time with automation increase which must be associated with skills increase at the time, then there is no more a dichotomy between the both variables since like stipulates assumption 3, the skilled labor marginal productivity remains constant over time.

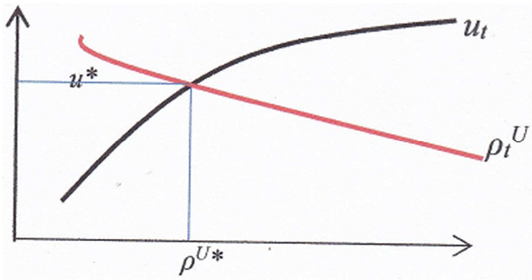

Figure 3. The Machines used unique Equilibrium.

In proposition 4 second part, if automation of tasks by machines and unskilled labor demand moves in a stochastic way, then there exist multiple equilibria in machines used rather than labor such that, if growth remains constant, then labor is not hurts by machines. Otherwise, if growth is stochastic, then machines are more used in production than labor. That part refers to figure 4 below.

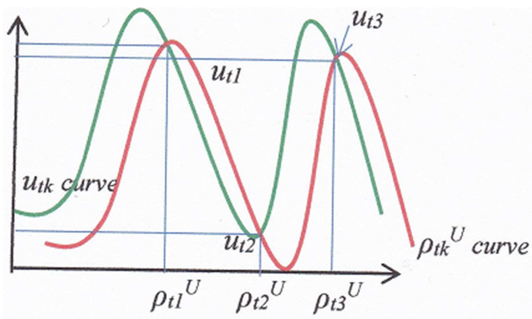

Figure 4. The Machine used Multiple Equilibria.

6. Conclusion

In this article, we have prolonged, *first* the work [9], *second,* [10] in introducing the unskilled labor inside the study of the role of capital-skill complementarity in skills decrease, thus in labor demand. In contrast to the basic models, since labor decrease caused by capita-skill complementarity or new innovations introduction in good production has an impact in the both kinds of labour i.e skilled and unskilled. Without introducing the skilled labor unemployment risk, national income factor shares equilibrium is unable to be properly established since poverty as well as inequalities in income might increase over time if labor heterogeneity in national income shares dilemma is not included in the study. When the skilled remains the major participant of the economic growth foundations, growth may keep increasing has shown by the literature, but evictions also yield raising social problems and reduces winners in national income and increase looser i.e poor agents more. Therefore, this article stipulates that, labor decrease empirically observed in national income shares caused by technological change, ruling out the equilibrium among national income factor shares, overcome through the balanced growth path establishment needs not only more human capital, but also learning in new technology while working for the equilibrium to exist, since skills decrease as well as the switch of engineers from production to research is slowed. Robotization or capital-skill complementarity in our article demanding more skills in the labour market is the concern of the unskilled labor through the hypothesis that, skilled labor marginal productivity is constant through the time. Consequently, the growth model developed here, admits a balanced growth path of two kinds, among capital and heterogenous labor shares in national income and between machines and humans, specifically, unskilled labor along with state variables move at the same constant rate near the steady allowing stability to hold on the two equilibria types. Finally, the factor shares unbalanced and instability in national income is solved through five concepts, perpetual youth [32, 33], human capital accumulation [34], learning by doing of new technology while working or trained [25] and capital-skill complementarity or input augmenting technology [10] and automation [11]. The second part of the study focuses on the relationship between machines and unskilled labor in order to establish a way for the both workers kinds to move at the same constant rate along the steady state, avoiding excess automation, thus social disorders and shows that, according to the threshold existence in machine used, equality among national income factor shares hold whenever, the BGP exist and is stable over time.

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