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Product Maximization Techniques of a Factory of Bangladesh: A Sustainable Procedure

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Abstract

Every factory wants to obtain a maximum production that provides a maximum profit, and also wants to go ahead with a sustainable way in the competitive global economy. To achieve a sustainable economic environment, the factory must run in ensuing scientific methods. This study has considered three inputs, such as capital, labor and other inputs for the sustainable production of the factory. In this article, the economic predictions are given by the comparative statics with detail mathematical analysis. Bangladesh is a developing country in the South Asia. It is densely populated country but short of skilled workforce. Agriculture is the principal sector of its economy. Only depending on agriculture sector a country cannot reach its economic peak. Recently, Bangladesh moves to industrialization. To increase national wealth and fortune, the country must produce high quality of products and needs to export surplus products in global markets. So, the production sector of the country has to increase production depending on the local and global demands. For a long-term sustainable production it must run the factories efficiently and future production techniques should be in scientific based. In this study an attempt has been taken to maximize products of a factory of Bangladesh subject to a budget constraint, using Lagrange multiplier technique, as well as, applying necessary and sufficient conditions to achieve optimal result.

Keywords

Bangladesh, Lagrange Multiplier, Product Maximization, Sustainability

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1. Introduction

Bangladesh is a developing country in the South Asia. Its official name is "The People's Republic of Bangladesh" (Figure 1). Its capital and the largest city is Dhaka, and Chittagong is the largest seaport. Its area is 148,460 km² and population in 2021 becomes more than 168,472,000. It is the 8th most populous country in the world. It borders with India to the west, north, and east (4,100 km), Myanmar to the southeast (247 km), and the Bay of Bengal to the south. It is one of the world's emerging and fastest-growing economies. Its life expectancy (73 years), literacy rates (74.7%), and per capita food production (181.3 kg per year) have increased significantly. But about 39 million people (23%) are still living below the national poverty line (per person income is $1.90 per day). The readymade garments (RMG) sector of Bangladesh has become the largest earner (about $15.54 billion per year) of foreign exchange. The minimum wage level in this country is Tk. 8,000 ($1= Tk. 85) per month [11].

Although agriculture is the leading sector of Bangladesh economy; recently, the country moves to industrialization. Mass industrial sector of the country are garments and textiles, leather and leather goods, electronics, light manufacturing, light engineering, energy and power, healthcare, information and communications technology (ICT), plastic, medical equipment, pharmaceutical, ship building, etc. Industrialization in the country is considered as

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the key to create new jobs that can reduce unemployment [23, 40]. Production in factories in developing country like Bangladesh is mostly labor intensive rather than industrial technology oriented, but from the last fifteen years innovative and energy efficient technology is increasing day by day. The industries ministry of Bangladesh has drafted the 5-year “National Industrial Policy 2016”. As a result, the new economic zones are being created by both public and private sectors [38].

In Bangladesh wage of labors are very low compared with most of the developing countries. The investment in industrial sector is not very satisfactory. Foreign direct investment ((FDI) is about $3.6 billion) and local investment are not hopeful due to political unrest and lack of continuous energy supply in the industrial sector. Other hindrances on industrialization in the country are inadequate infrastructure, limited financing instruments, bureaucratic delays, low enforcement of labor laws, etc. Bank loans are not sufficient to grow more factories. There is no sufficient worker rights and factory safety in the country. On the other hand, some small and medium enterprises (SMEs) are working efficiently in the country and some of them are exporting their valuable and demandable products. There are huge safety and continuous energy supply in the Export Processing Zones (EPZs) [12, 40].

Figure 1. Map of Bangladseh. Source: [11].

In this study we have tried to represent a suitable interpretation of the Lagrange multiplier in the context of product maximization of a factory of Bangladesh. We have also tried to show that it is a device for transforming a constrained problem to a higher dimensional unconstrained problem. We have developed the mathematical procedures where it indicates the shadow price and we observe that its value is positive. The factory must be sustainable, if it maintains all the procedures of optimum level for all inputs in production, and if it takes necessary attempts to prevent environment pollution [25, 37]. We have also used the Cobb-Douglas production function as a tool of economic analysis that helps the factory to achieve a maximum production. It satisfies the basic economic laws and interprets the estimated parameters that we have used [29].

We have also tried to find a relation between production function and Lagrange multiplier. In the study we have shown mathematical calculations in some details and have introduced theorems where necessary to make the article interesting to the common readers. We not only prove that the production of the factory is a maximum but also verify that it is really a maximum by using sufficient conditions. Finally, we have tried to predict the effects of the inputs for the maximum production so that the factory authority can take correct decision to choose raw materials, such as capital, labor, and other inputs.

2. Literature Review

The first empirical analysis of production functions, a relationship between inputs and outputs, was given by two American scholars, mathematician Charles W. Cobb and economist Paul H. Douglas in 1928. They had studied the functional distribution of income between capital and labor in the context of an aggregate macroeconomic production function [10]. Other two American researchers, mathematician John V. Baxley and economist John C. Moorhouse, have considered implicit functions with assumed characteristic qualitative features and have provided illustration of an example by generating meaningful economic behavior. They have also discussed aspects of production functions with sufficient mathematical techniques [4].

Pahlaj Moolio and his coauthors have considered the Cobb-Douglas function in three variables as an explicit form of production function. They have tried to maximize an output subject to a budget constraint, using Lagrange multipliers technique, as well as, necessary and sufficient conditions for optimal value have been applied [31]. Haradhan Kumar Mohajan and his coauthors have considered the optimization techniques and social welfare economics. They have discussed mathematical economics and social choices for the welfare of the society [23].

Ana M. Fernandes studies the total factor productivity for manufacturing firms in Bangladesh by analyzing 575 firms of the country [12]. Industrial policy mainly emphasizes on engagement of mechanism and stimulation to increase the productivity for maximum returns from capital and labor [1]. Sodip Roy has stressed on green growth, green energy, and green industrialization for economic development in Bangladesh. He also realizes that the Government of Bangladesh has formulated Industrial Policy 2016, which has been embedded with several targets conducive to green growth and green production [38].

In a research paper, Lia Roy and her coauthors have considered cost minimization of a running industry by using Cobb-Douglas production function and taking three variables capital, labor, and other inputs for the research analysis. Their mathematical modeling in economics provides the meaningful economic behavior for the sustainability of an industry [37]. Haradhan Kumar Mohajan has tried to explain the Cobb-Douglas production function, by statistical analysis, to predict the cost minimization policies of a running garments industry of Bangladesh [29].

3. Research Methodology of the Study

Research is an essential and influential device to lead a researcher towards proper academic development [34]. Methodology is the guidelines to complete the activities efficiently and accurately. Research methodology provides us the effective principles for planning, arranging, designing and conducting a fruitful research. Hence, we can consider it as a pioneer path with the application of science and philosophy to perform all researches confidently [19].

In this paper we have used secondary data. The data have been collected from both published and unpublished data sources, such as reputed journals, books of well-known authors, conference proceedings, submitted and preprint papers, various publications of national and international organizations, websites, handbooks, theses, various research reports, newspapers, information on internet, etc.

In this study we have introduced some basic definitions of mathematical economics that are related to our article, for the convenience of new researchers. Then we step to find a relation between production function and Lagrange multiplier μ; using K quantity of capital, L quantity of labor, and R quantity of other inputs for the production of M units. We have used the Cobb-Douglas production function, $F(K, L, R) = AK^x L^y R^z$, for the sustainable production, where the symbols have their usual meanings. In the study we have introduced some theorems and lemmas and illustrative results to make the article interesting to the readers. We have also test the comparative statics, such as $\frac{\partial K}{\partial r}, \frac{\partial L}{\partial r}, \frac{\partial K}{\partial B}$, etc. to predict the production procedures by the changes in the costs of capital, labor, and other inputs with the economic analysis. In the study we have tried to find the optimization with necessary and sufficient conditions. Throughout the study we have tried to provide the mathematical calculations with some details. To perform a satisfactory research we have to remember that reliability and validity are vital elements. In every step of our research procedure we were very careful and tried to maintain them as long as possible [25, 26, 28].

4. Objective of the Study

The main objective of this study is to obtain maximum production of a factory of Bangladesh using various inputs by a scientific method. As a result, all the factories of the country will run in sustainable ways. The other minor objectives are as follows:

a) to give mathematical procedures very clearly and accurately,

b) to present the model with the interpretation of Lagrange multiplier, and

c) to provide economic analysis with necessary and sufficient conditions.

5. Some Elementary Definitions

Mathematical economics is a research area of combination of mathematics and economics. Here the researchers need vast knowledge from both mathematics and economics. Sometimes a researcher has deep knowledge in mathematics but light in economics, and vice versa. We think, our article related some basic definitions of mathematical economics will be helpful for the common readers. Therefore, we have included some basic definitions those who are novice in this field. Most of the expert researchers can skip this section and can go through the main text.

5.1. Exogenous and Endogenous Variables

Some economic variables are determined by our mathematical models and others are usually assumed to be determined by the factors outside of our models, i.e., not to be determined by the economist. In a mathematical model, the value of an exogenous variable is determined from outside of the model and is included on the model. An exogenous change is made by changing an exogenous variable. Exogenous comes from the two Greek words: exo- which means "outside" and gignomai, which means "to produce". It is similar to an independent variable and sometimes called predetermined variable [42]. On the other hand, if the value of a variable is determined jointly by the particular values taken by the exogenous variables and by the logical relationships among variables within the model, is called endogenous variable. Endogenous come from two Greek words: endo- meaning "inside" and genous, meaning "producing". An endogenous change is happened if there is a change in an endogenous variable in response to an exogenous change that is imposed upon the model. It is similar to a dependent variable because, it is influenced by one or more independent variables and sometimes called jointly determined variable [20]. Exogenous variables are thought of as causes, whereas endogenous as their effects. Generally, inputs in the model are exogenous variables; the price of production is also exogenous variable. For example, in farming, variables, such as weather, farmer skill, soil fertility, pests, diseases and availability of seed and water are all exogenous to crop production; and the production of crops is endogenous variable. Exogenous variables always act to explain the variations in endogenous variables. Exogenous variables influence the endogenous variables but are not themselves influenced by them. One variable which is endogenous for one model may be exogenous for the other model [35].

5.2. Hessian

In mathematics, the Hessian matrix or Hessian is a square matrix that contains second-order partial and cross-partial derivatives of a scalar function. It tries to arrange the local curvature of a function of many variables. It was first established in the 19th century by the German mathematician Ludwig Otto Hesse (1811–1874), and later it is named after him. It is a symmetric matrix [8]. Let us suppose an n-dimensional function $f: R^n \to R$, where $\text{x} = (x_1, x_2, \dots, x_n) \in R^n$ and $f(\text{x}) \in R$. If all second order partial derivatives of f exist and are continuous over the domain of the function, then the Hessian matrix H of f is a square $n \times n$ matrix, usually defined and arranged as [9];

$$H = \begin{bmatrix} f_{x_1^2} & f_{x_1 x_2} & \cdots & f_{x_1 x_n} \\ f_{x_2 x_1} & f_{x_2^2} & \cdots & f_{x_2 x_n} \\ \cdots & \cdots & \cdots & \\ f_{x_n x_1} & f_{x_n x_2} & \cdots & f_{x_n^2} \end{bmatrix}. \quad (1)$$

The Hessian is used to describe stationary points of unconstrained optimization problems, which are drawn from the theory of the firm. For example, a firm wants to produce M units of product by using capital K, and labor L, by using the production function $f(K, L)$. The firm wants to maximize its profit and can use Hessian to reach its goal. For $|H| < 0$, the marginal product of labor must be diminishing; additional labor beyond the optimal choice must decrease productivity, and also decreases profit. For $|H| > 0$, the marginal product of capital must also be diminishing. If a firm has no sufficient knowledge of the nature of Hessian, the firm will be unsustainable for inefficient use of its resources and a decrease is visible in the production. Hence, the Hessian matrix plays an important tool in the policy analysis of unconstrained choices [3].

5.3. Jacobian

In vector calculus, the Jacobian matrix is consists of by taking several variables of a vector-valued function and all its elements are first-order partial derivatives. Both the matrix and the determinant of such types are considered as the Jacobian. It was established by a German Jewish mathematician, Carl Gustav Jacob Jacobi (1804-1851). Let us consider $F: R^n \to R^m$, where $\text{x} = (x_1, x_2, \dots, x_n) \in R^n$ and $F(\text{x}) \in R^m$ is a function such that each of its first-order partial derivatives exists on R^n. The Jacobian matrix of F is defined to be an $m \times n$ matrix and is denoted by [36],

$$J = \frac{\partial(F_1, F_2, \dots, F_m)}{\partial(x_1, x_2, \dots, x_n)} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}. \quad (2)$$

The Jacobian is important in many branches of mathematics, because if the function F is differentiable at a point x, then the Jacobian matrix defines a linear map, $F: R^n \to R^m$, which is the best linear approximation of the function F near the point x [8]. The Jacobian is usually the determinant of this matrix when the square matrix is of order $n \times n$ and is denoted by,

$$|J| = \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{vmatrix}. \quad (3)$$

5.4. Necessary and Sufficient Conditions of Optimization

Every factory wants to maximize its profit and revenue, and also wants to minimize its cost; which is simply called optimization. There are two conditions on optimization: i) necessary condition, and ii) sufficient condition.

5.4.1. Necessary Condition

Let us consider a function *f*(x) of single variable x, where $x=(x_1,x_2,...,x_n)$. For a function *f*(*x*) to be optimum (maximum or minimum) $\frac{df}{dx}=f'(x)=0$. If $f(x,y)$ be a function of two variables x and y then for optimum, $\frac{\partial f}{\partial x}(i.e.f_x)=0=\frac{\partial f}{\partial y}$ (i.e., f_y). This is called necessary condition [27].

5.4.2. Sufficient Condition

If a function $f(x)$ of single variable x, then if $\frac{d^2f}{dx^2}<0$ at $x=a$ the function $f(x)$ is maximum at that point $x=a$, and if $\frac{d^2f}{dx^2}>0$ at $x=a$ the function $f(x)$ is minimum at that point $x=a$. If $f(x,y)$ be a function of two variables x and y then $D=f_{xx}f_{yy}-f_{xy}^2>0$. If $f_{xx}>0$ (and $f_{yy}>0$), then the function has a minimum point (x_0,y_0), if $f_{xx}<0$ (and $f_{yy}<0$) then the function has a maximum point (x_0,y_0). For $D=f_{xx}f_{yy}-f_{xy}^2<0$, there is neither a maximum nor a minimum, but a saddle point. In all cases, the tangent plane at the extremum (maximum or minimum) or a saddle point to the surface $z=f(x,y)$, is parallel to the *z*-plane. If $D=f_{xx}f_{yy}-f_{xy}^2=0$, one has to apply other considerations to determine the nature of the extremum [27].

5.5. Constrained Optimization

In mathematical optimization, constrained optimization is considered as the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables. The objective function is either a cost or energy function, which is to be minimized; a profit or reward or revenue or utility function, which is to be maximized [41].

A general constrained minimization problem may be written as follows:

Maximum

$$z=f(x_1,x_2), \tag{4}$$

subject to,

$$G_\alpha(x_1,x_2)=A_\alpha \text{ for } \alpha=1,2,...,n;$$

which are equality constraints, and

$$H_\beta(x_1,x_2)=A_\beta \text{ for } \beta=1,2,...,n; \tag{5}$$

which are inequality constraints. One significant constraint in every problem, it is either maximum or minimum, is its budget constraint. In real life, producers face several economic constraints, such as resource constraints, production constraints, etc. Here $f:R^n\to R$ is called the objective function. Constrained optimization problems play an important role in the development of algorithms and software for the general constrained problem because; many algorithms reduce the solution of the general problem to the solution of a sequence of bound-constrained problems [6].

5.6. Unconstrained Optimization

An unconstrained problem is a function in the case when there are no constraints on its arguments. Unconstrained optimization tries to find the maximum or minimum values of a differentiable function of several variables over a nice set. At the stationary points, the gradient of a function is zero; at a local minimum the Hessian is positive definite, at a local maximum the Hessian is negative definite, and at other stationary points, such as at saddle points, the Hessian is indefinite [22].

Unconstrained optimization problems consider the problem of optimization of an objective function that depends on real variables with no restrictions on their values. Mathematically, let $x\in R^n$ be a real vector with $n\geq 1$ components and let $f:R^n\to R$ be a smooth function. Then, the unconstrained optimization problem is,

$$\max/\min f(x), \tag{6}$$

where $f(x)$ is convex and twice differentiable in every direction. For real-valued functions of two variables, an unconstrained optimization problem is [7],

$$\max/\min f(x,y). \tag{7}$$

The function $f(x)$ has a gradient vector $\nabla f(x)$, and is defined at every point *x as,*

$$\nabla f(x) = \begin{vmatrix} f_{x_1} \\ f_{x_2} \\ \cdots \\ f_{x_n} \end{vmatrix}. \quad (8)$$

The Hessian matrix $\nabla^2 f(x)$, which is symmetric and is defined at every point *x as,*

$$|H| = |\nabla^2 f(x)| = \begin{vmatrix} f_{x_1^2} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2^2} & \cdots & f_{x_2x_n} \\ \cdots & \cdots & \cdots & \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_n^2} \end{vmatrix} \quad (9)$$

where Hessian $|H| > 0$ for minimum and $|H| < 0$ for maximum [Boyd & Vandenberghe, 2009]. Unconstrained optimization problems come directly in some applications but they also come indirectly from reformulations of constrained optimization problems. Often it is practical to replace the constraints of an optimization problem with penalized terms in the objective function and to solve the problem as an unconstrained problem [33].

5.7. Lagrange Multipliers

In mathematical and economical optimization, the method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality. It is named after the Italian mathematician and astronomer, Joseph-Louis Lagrange (1736-1813) [14]. It is used to convert a constrained problem into a higher dimensional unconstrained problem. The relationship between the gradient of the function and gradients of the constraints rather naturally leads to a reformulation of the original problem, known as the Lagrangian function [5]. Let us consider a function of two variables [9],

$$\text{maximize } z = f(x_1, x_2) \quad (10)$$

subjected to a single constraint,

$$g(x_1, x_2) = k. \quad (11)$$

We introduce a new variable μ, called a Lagrange multiplier and introduce the Lagrange function as,

$$u(x_1, x_2, \mu) = f(x_1, x_2) + \mu\{g(x_1, x_2) - k\}. \quad (12)$$

For optimization,

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial \mu} = 0. \quad (13)$$

For multiple constraints [2],

$g_1(x_1, x_2) = k_1$, $g_2(x_1, x_2) = k_2$, … … …, $g_n(x_1, x_2) = k_n$, the Lagrange function can be written as,

$$u(x_1, x_2, \mu) = f(x_1, x_2) + \mu_1\{g_1(x_1, x_2) - k_1\} + \mu_2\{g_2(x_1, x_2) - k_2\} + \ldots + \mu_n\{g_n(x_1, x_2) - k_n\}, \quad (14)$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are Lagrange multipliers.

5.8. Implicit Function Theorem

In multivariable calculus, the implicit function theorem is a device that allows relations to be converted to functions of several real variables. Augustin-Louis Cauchy (1789-1857), a French mathematician, engineer, and physicist, for the first time provides strenuous form of the implicit function theorem [18]. To state the implicit function theorem, we need the Jacobian matrix (3) of *G*, which is the matrix of the partial derivatives of *G*. Let us suppose, $G(x_0, c_0) = 0$ and $\frac{\partial G(x_0, c_0)}{\partial x} \neq 0$. Then there exists a continuous implicit solution $x(c)$, where *c* is some parameter, with derivative,

$$\frac{\partial x(c)}{\partial x} = -\frac{G_c(x(c), c)}{G_x(x(c), c)} \quad (15)$$

for *c* close to c_0 [23, 32].

5.9. Comparative Static Analysis

In economics, comparative statics is the comparison of two different economic outcomes, before and after a change in some underlying exogenous parameter [21]. Comparative statics was formalized by British economist, John R. Hicks (1904-1989) and American economist, Paul A. Samuelson (1915-2009) [13, 39].

In the society the behavior of the buyers and sellers often changes, which causes the change of demand and supply curves to itself over time. In economics, it is important to analyze how these changes affect equilibrium. Mathematically, we can write twelve partial derivatives in a matrix as [25],

$$\begin{bmatrix} X_{P_1} & Y_{P_1} & L_{P_1} & \mu_{P_1} \\ X_{P_2} & Y_{P_2} & L_{P_2} & \mu_{P_2} \\ X_w & Y_w & L_w & \mu_w \end{bmatrix}. \quad (16)$$

Here *X* and *Y* are two commodities and *L* indicates total labors, and μ is Lagrange multiplier. Moreover, P_1 and P_1 are the prices of per unit commodities X and Y respectively, and w is wage rate. The twelve partial derivatives in matrix (16) are called the *comparative statics* of the problem [9]. For example, if P be the price of a commodity X, then $X_P = \frac{\partial X}{\partial P} < 0$ indicates that if the price of commodity X increases, the level of consumption of X will decrease [15, 16, 27].

5.10. Shadow Price

A shadow price is a monetary value allocated to currently unknowable or difficult to calculate costs in the absence of correct market prices. It is an estimated price for something that is not normally priced in the market or sold in the market. It is often calculated based on certain assumptions, and so it is subjective and somewhat inaccurate. It is the real economic price of projects, activities, goods, and services that have no market price [17]. In economics, the shadow price of a commodity is defined as its social opportunity cost, i.e., the net loss (gain) associated with having 1 unit less (more) of it. For example, if $\frac{\partial K}{\partial M} = \mu$, then if the firm wants to increase (decrease) 1 unit of its production, it would cause total capital to increase (decrease) by approximately μ units [15, 16, 27].

6. Relation Between Production Function and Lagrange Multiplier

Let us consider a factory of Bangladesh wants to produce and provide M units of products in a year using K quantity of capital, L quantity of labor, and R quantity of other inputs for the welfare of citizens of the country. Here K, R, and L are exogenous variables. The objective of the factory is to maximize the product function,

$$M = F(K, L, R), \tag{17}$$

subject to the budget constraint,

$$B = rK + wL + \rho R, \tag{18}$$

where r is rate of interest or services of capital per unit of capital K; w is the wage rate per unit of labor L; and ρ is the cost per unit of other inputs R; while F is a suitable production function. We assume that second order partial derivatives of the function F with respect to the independent variables (factors) K, L, and R exist [15, 16, 27].

Now we introduce a single Lagrange multiplier μ in (17) and (18), and define the Lagrangian function V, in a four dimensional unconstrained problem as follows:

$$V(K, L, R, \mu) = M(K, L, R) + \mu(B - rK - wL - \rho R). \tag{19}$$

Equation (19) is a four-dimensional unconstrained problem obtained from equations (17) and (18) by the use of Lagrange multiplier μ, as a device. We assume that the industry maximizes its products, the optimal quantities K^*, L^*, R^*, and μ^* of K, L, R, and μ, respectively that necessarily satisfy the first order conditions; which we obtain by partial differentiation of the Lagrangian function (19) with respect to four variables K, L, R, and μ; and setting them equal to zero we obtain,

$$\begin{aligned} V_\mu &= B - rK - wL - \rho R = 0, \\ V_K &= M_K - \mu r = 0, \\ V_L &= M_L - \mu w = 0, \\ V_R &= M_R - \mu\rho = 0, \end{aligned} \tag{20}$$

where $V_K = \frac{\partial V}{\partial K}$, etc. are partial derivatives. Now we want to find a relationship of Lagrange multiplier μ with the product function. In the following theorem we will try to search the effects of future production of a factory when its budget changes [23].

Theorem 1: Prove that the Lagrange multiplier μ can be interpreted as the marginal production as, $\frac{dM}{dB} = \mu$.

Proof: Let us consider that a factory of Bangladesh wants to produce M units of its production using K quantity of capital, L quantity of labor, and R quantity of other inputs. The factory has total budget B units to run the factory smoothly. Now we use the optimization techniques of (20) to prove this theorem.

From the second equation of (20) we get,

$$\frac{M_K}{r} = \mu.$$

From the third equation of (20) we get,

$$\frac{M_L}{w} = \mu. \tag{21}$$

From the fourth equation of (20) we get,

$$\frac{M_R}{\rho} = \mu.$$

Combining (21) we find the Lagrange multiplier as;

$$\frac{M_K}{r} = \frac{M_L}{w} = \frac{M_R}{\rho} = \mu. \tag{22}$$

Considering the infinitesimal changes dK, dL, dR in K, L, R, respectively, and the corresponding changes dM and dB in M

and B, respectively, we get;

$$dM = M_K dK + M_L dL + M_R dR\,, \tag{23}$$

$$dB = rdK + wdL + \rho dR\,. \tag{24}$$

Dividing (23) by (24) we get,

$$\frac{dM}{dB} = \frac{M_K dK + M_L dL + M_R dR}{rdK + wdL + \rho dR}\,. \tag{25}$$

If L and R remain constants, K varies then $dL = 0$ and $dR = 0$, hence (25) takes the form,

$$\frac{dM}{dB} = \frac{M_K dK}{rdK} = \frac{M_K}{r} = \mu\,.$$

If K and R remain constants, L varies then $dK = 0$ and $dR = 0$, hence (25) takes the form,

$$\frac{dM}{dB} = \frac{M_L dL}{wdL} = \frac{M_L}{w} = \mu\,. \tag{26}$$

If K and L remain constants, R varies then $dK = 0$ and $dL = 0$, hence (23) takes the form,

$$\frac{dM}{dB} = \frac{M_R dR}{\rho dR} = \frac{M_R}{\rho} = \mu\,.$$

Combining (26) we obtain,

$$\frac{dM}{dB} = \frac{M_K}{r} = \frac{M_L}{w} = \frac{M_R}{\rho} = \mu\,. \tag{27}$$

Equations (25) and (27) can be written as,

$$\frac{dM}{dB} = \mu\,. \tag{28}$$

Thus, the theorem is proved.

From (28) we see that the Lagrange multiplier can be interpreted as the marginal production. It indicates that total production, M will be increased μ units from an additional unit of budget, B. If the factory wants to increase (decrease) a unit of its production, it must increase (decrease) exactly $1/\mu$ units of its budget B [23, 37].

7. An Economic Example of Cobb-Douglas Production Function

Let us consider the Cobb-Douglas production function F is given by [10],

$$M = F(K, L, R) = AK^x L^y R^z\,, \tag{29}$$

where A is the efficiency parameter reflecting the level of technology, i.e., technical process, economic system, etc., which represents total factor productivity. Moreover, A also reflects the skill and education level of the workforce. Here x, y, and z are constants; x indicates the output of elasticity of capital measures the percentage change in M for 1% change in K, while R and L are held constants; y indicates the output of elasticity of labor, and z indicates the output of elasticity of other inputs in the production process, are exactly parallel to x. The values of x, y, and z are determined by available technologies. Now these three constants x, y, and z must satisfy the following three inequalities [29, 37]:

$$0 < x < 1,\ 0 < y < 1,\ \text{and}\ 0 < z < 1\,. \tag{30}$$

A strict Cobb-Douglass production function, in which $x + y + z < 1$ indicates decreasing returns to scale, $x + y + z = 1$ indicates constant returns to scale, and $x + y + z > 1$ indicates increasing returns to scale.

Now using (20), (22), and (29) in (24) we get [15, 16, 31],

$$V(K, L, R, \mu) = AK^x L^y R^z + \mu(B - rK - wL - \rho R)\,. \tag{31}$$

For maximization first order differentiation equals to zero, then from (31) we can write,

$$\begin{aligned} V_\mu &= B - rK - wL - \rho R = 0\,, \\ V_K &= xAK^{x-1} L^y R^z - r\mu = 0\,, \\ V_L &= yAK^x L^{y-1} R^z - w\mu = 0\,, \\ V_R &= zAK^x L^y R^{z-1} - \rho\mu = 0\,. \end{aligned} \tag{32}$$

From the first equation of (32)) we get,

$$B = rK + wL + \rho R\,. \tag{33}$$

In the following theorem we will try to prove, using Cobb-Douglas production, a factory can able to achieve maximum production. For the sustainability of the factory, it must invest various inputs properly and efficiently. It must adjust all the parameters of production with the changing situation at present and in future. In the following Theorem 2, we use equations (30) to (33) [30, 31].

Theorem 2: A factory can produce M units of products in a year using K quantity of capital, L quantity of labor, and R quantity of other inputs. Using Cobb-Douglas production function $M = F(K, L, R) = AK^x L^y R^z$ prove that Lagrange multiplier can be expressed as, $\mu = \dfrac{Ax^x y^y z^z B^{\Gamma-1}}{r^x w^y \rho^z \Gamma^{\Gamma-1}}$, and the maximum production function of the factory is,

$M = \frac{Ax^x y^y z^z B^\Gamma}{r^x w^y \rho^z \Gamma^\Gamma}$, where $\Gamma = x + y + z$ and B is the budget of the factory.

Proof: The factory uses K quantity of capital, L quantity of labor, and R quantity of other inputs within its budget B. To prove the theorem we use the techniques of equations (31), (32) and (33). From the second equation of (32) we get,

$$\mu = \frac{xAK^x L^y R^z}{rK}$$

$$\Rightarrow rK = \frac{xAK^x L^y R^z}{\mu}. \quad (34)$$

From the third equation of (32) we get,

$$\mu = \frac{yAK^x L^y R^z}{wL}$$

$$\Rightarrow wL = \frac{yAK^x L^y R^z}{\mu}. \quad (35)$$

From the fourth equation of (32) we get,

$$\mu = \frac{zAK^x L^y R^z}{\rho R}$$

$$\Rightarrow \rho R = \frac{cAK^a L^b R^c}{\mu}. \quad (36)$$

Substituting the values from (34), (35), and (36) in (33) we get,

$$B = \frac{xAK^x L^y R^z}{\mu} + \frac{yAK^x L^y R^z}{\mu} + \frac{zAK^x L^y R^z}{\mu}$$

$$B = \frac{AK^x L^y R^z}{\mu}(x + y + z)$$

$$B = \frac{AK^x L^y R^z \Gamma}{\mu} \quad (37)$$

where $\Gamma = x + y + z$.

Putting the value of μ from the first equation of (34) in (37) we get,

$$B = \frac{AK^x L^y R^z \Gamma}{\frac{xAK^x L^y R^z}{rK}}$$

$$B = \frac{rK\Gamma}{x}.$$

$$\Rightarrow K = K^* = \frac{xB}{r\Gamma} \quad (38)$$

where $\Gamma = x + y + z$.

Putting the value of μ from the first equation of (35) in (37) we get,

$$B = \frac{AK^x L^{by} R^z \Gamma}{\frac{yAK^x L^y R^z}{wL}}$$

$$B = \frac{wL\Gamma}{y}.$$

$$\Rightarrow L = L^* = \frac{yB}{w\Gamma}. \quad (39)$$

Putting the value of from the first equation of (36) in (37) we get,

$$B = \frac{AK^x L^y R^z \Gamma}{\frac{zAK^x L^y R^z}{\rho R}}$$

$$B = \frac{\rho R\Gamma}{z}.$$

$$\Rightarrow R = R^* = \frac{zB}{\rho\Gamma}. \quad (40)$$

The stationary point for the production can be written as;

$$\left(K^*, L^*, R^*\right) = \left(\frac{xB}{r\Gamma}, \frac{yB}{w\Gamma}, \frac{zB}{\rho\Gamma}\right). \quad (41)$$

Putting the values of K, L, and R from (38), (39), and (40) in (34) we get,

$$\mu = \frac{xA\left(\frac{xB}{r\Gamma}\right)^x \left(\frac{yB}{w\Gamma}\right)^y \left(\frac{zB}{\rho\Gamma}\right)^z}{\frac{xB}{\Gamma}}$$

$$= xA \times \frac{x^x B^x}{r^x \Gamma^x} \times \frac{y^y B^y}{w^y \Gamma^y} \times \frac{z^z B^z}{\rho^z \Gamma^z} \times \frac{\Gamma}{xB}$$

$$\mu = \mu^* = \frac{Ax^x y^y z^z B^{\Gamma-1}}{r^x w^y \rho^z \Gamma^{\Gamma-1}}. \quad (42)$$

Now substituting the values of K, L, and R from (38), (39), and (40) into (29), we get the optimal value of the production function,

$$M = A\left(\frac{xB}{r\Gamma}\right)^{x}\left(\frac{yB}{w\Gamma}\right)^{y}\left(\frac{zB}{\rho\Gamma}\right)^{z}$$

$$\Rightarrow M = \frac{Ax^{x}y^{y}z^{z}B^{\Gamma}}{r^{x}w^{y}\rho^{z}\Gamma^{\Gamma}}. \quad (43)$$

Hence, the theorem is proved.

Equation (43) is the production function in terms of $r, w, \rho, A, B > 0$, $x, y, z > 0$, and $\Gamma = x + y + z > 0$. All the parameters in right hand side of (43) are known to the factory authority and can easily calculate its maximum production. Now we want to discuss the results of Theorem 2 by three Lemmas and three arithmetic results as follows:

Lemma 1: If $x = y = \frac{1}{4}$ and $z = \frac{1}{2}$, in constant returns scale prove that, $\mu = \frac{A\left(\frac{1}{8}\right)^{\frac{1}{2}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{2}}}$ and $M = \frac{AB\left(\frac{1}{8}\right)^{\frac{1}{2}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{2}}}$.

Proof: For constant returns scale let us consider, $x = y = \frac{1}{4}$ and $z = \frac{1}{2}$, then $\Gamma = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$. Putting the values in (42) we get,

$$\mu = \frac{A\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{1}{2}\right)^{\frac{1}{2}}B^{0}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{2}} \times 1^{0}}$$

$$= \frac{A\left(\frac{1}{8}\right)^{\frac{1}{2}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{2}}}. \quad (44)$$

Putting the above values in (43) we get,

$$M = \frac{A\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{1}{2}\right)^{\frac{1}{2}}B}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{2}}}$$

$$M = \frac{AB\left(\frac{1}{8}\right)^{\frac{1}{2}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{2}}}. \quad (45)$$

Hence, Lemma 1 is proved.

Result 1: In Lemma 1 if $A = 3, B = \$1,000,0000$, $r = w = 9$, and $\rho = 4$, then from (44) we can calculate the Lagrange multiplier as;

$$\mu = \frac{3\left(\frac{1}{8}\right)^{\frac{1}{2}}}{(9)^{\frac{1}{4}}(9)^{\frac{1}{4}}(4)^{\frac{1}{2}}} = \frac{3\left(\frac{1}{8}\right)^{\frac{1}{2}}}{3 \times 2} = \$0.18. \quad (46)$$

From (45) we can calculate the production of the factory as,

$$M = \frac{3 \times 1,000,0000 \times \left(\frac{1}{8}\right)^{\frac{1}{2}}}{(9)^{\frac{1}{4}}(9)^{\frac{1}{4}}(4)^{\frac{1}{2}}}$$

$$M = 500,0000 \times \left(\frac{1}{8}\right)^{\frac{1}{2}} = 176,777 \text{ units.} \quad (47)$$

Lemma 2: If $x = y = \frac{1}{3}$ and $z = \frac{1}{2}$, in increasing returns scale prove that, $\mu = \frac{A\left(\frac{1}{3}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}B^{\frac{1}{6}}}{(r)^{\frac{1}{3}}(w)^{\frac{1}{3}}(\rho)^{\frac{1}{2}}\left(\frac{7}{6}\right)^{\frac{1}{6}}}$ and

$$M = \frac{A\left(\frac{1}{3}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}B^{\frac{7}{6}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{2}}\left(\frac{7}{6}\right)^{\frac{7}{6}}}.$$

Proof: For increasing returns scale let us consider, $x = y = \frac{1}{3}$ and $z = \frac{1}{2}$, then $\Gamma = \frac{1}{3} + \frac{1}{3} + \frac{1}{2} = \frac{7}{6}$. Putting the values in (42) we get the Lagrange multiplier as,

$$\mu = \frac{A\left(\frac{1}{3}\right)^{\frac{1}{3}}\left(\frac{1}{3}\right)^{\frac{1}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}B^{\frac{1}{6}}}{(r)^{\frac{1}{3}}(w)^{\frac{1}{3}}(\rho)^{\frac{1}{2}} \times \left(\frac{7}{6}\right)^{\frac{1}{6}}}$$

$$= \frac{A\left(\frac{1}{3}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}B^{\frac{1}{6}}}{(r)^{\frac{1}{3}}(w)^{\frac{1}{3}}(\rho)^{\frac{1}{2}}\left(\frac{7}{6}\right)^{\frac{1}{6}}}. \quad (48)$$

Putting the above values in (43) we get the production of the factory as,

$$M=\frac{A\left(\frac{1}{3}\right)^{\frac{1}{3}}\left(\frac{1}{3}\right)^{\frac{1}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}B^{\frac{7}{6}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{2}}\left(\frac{7}{6}\right)^{\frac{7}{6}}}$$

$$M=\frac{A\left(\frac{1}{3}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}B^{\frac{7}{6}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{2}}\left(\frac{7}{6}\right)^{\frac{7}{6}}}. \qquad (49)$$

Hence, Lemma 2 is proved.

Result 2: For increasing returns scale in Lemma 2, if $A=3, B=\$1,000,0000$, $r=w=9$, and $\rho=4$, then from (48) we can be calculate the Lagrange multiplier as,

$$\mu=\frac{3\left(\frac{1}{3}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\times(1{,}000{,}0000)^{\frac{1}{6}}}{(9)^{\frac{1}{4}}(9)^{\frac{1}{4}}(4)^{\frac{1}{2}}\left(\frac{7}{6}\right)^{\frac{1}{6}}}$$

$$=\frac{3\left(\frac{1}{3}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\times(1{,}000{,}0000)^{\frac{1}{6}}}{3\times2\times\left(\frac{7}{6}\right)^{\frac{1}{6}}}=\$1.66\,. \qquad (50)$$

From (49) we can find the total product as;

$$M=\frac{3\left(\frac{1}{3}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\times(1{,}000{,}0000)^{\frac{7}{6}}}{(9)^{\frac{1}{3}}(9)^{\frac{1}{3}}(4)^{\frac{1}{2}}\left(\frac{7}{6}\right)^{\frac{7}{6}}}$$

$$M=\frac{\left(\frac{1}{3}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\frac{1}{2}}\times(1{,}000{,}0000)^{\frac{7}{6}}}{2\times\left(\frac{7}{6}\right)^{\frac{7}{6}}}=1{,}419{,}939 \text{ units.} \qquad (51)$$

Lemma 3: If $x=y=\frac{1}{4}$ and $z=\frac{1}{8}$, in decreasing returns scale prove that, $\mu=\frac{A\left(\frac{1}{4}\right)^{\frac{1}{2}}\left(\frac{1}{8}\right)^{\frac{1}{8}}\left(\frac{5}{8}\right)^{\frac{3}{8}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{8}}B^{\frac{3}{8}}}$ and

$$M=\frac{A\left(\frac{1}{4}\right)^{\frac{1}{2}}\left(\frac{1}{8}\right)^{\frac{1}{8}}B^{\frac{5}{8}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{8}}\left(\frac{5}{8}\right)^{\frac{5}{8}}}.$$

Proof: For decreasing returns scale let us consider, $x=y=\frac{1}{4}$ and $z=\frac{1}{8}$, then $\Gamma=\frac{1}{4}+\frac{1}{4}+\frac{1}{8}=\frac{5}{8}$. Putting the values in (43) we get the Lagrange multiplier as,

$$\mu=\frac{A\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{1}{8}\right)^{\frac{1}{8}}B^{-\frac{3}{8}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{8}}\times\left(\frac{5}{8}\right)^{-\frac{3}{8}}}$$

$$=\frac{A\left(\frac{1}{4}\right)^{\frac{1}{2}}\left(\frac{1}{8}\right)^{\frac{1}{8}}\left(\frac{5}{8}\right)^{\frac{3}{8}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{8}}B^{\frac{3}{8}}}. \qquad (52)$$

From (43) we get the production of the factory as,

$$M=\frac{A\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{1}{8}\right)^{\frac{1}{8}}B^{\frac{5}{8}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{8}}\left(\frac{5}{8}\right)^{\frac{5}{8}}}$$

$$M=\frac{A\left(\frac{1}{4}\right)^{\frac{1}{2}}\left(\frac{1}{8}\right)^{\frac{1}{8}}B^{\frac{5}{8}}}{(r)^{\frac{1}{4}}(w)^{\frac{1}{4}}(\rho)^{\frac{1}{8}}\left(\frac{5}{8}\right)^{\frac{5}{8}}}. \qquad (53)$$

Hence, the Lemma 3 is proved.

Result 3: For decreasing returns scale in Lemma 1 if $A=3, B=\$1,000,0000$, $r=w=9$, and $\rho=4$, then from (52) we get the Lagrange multiplier as,

$$\mu=\frac{3\left(\frac{1}{4}\right)^{\frac{1}{2}}\left(\frac{1}{8}\right)^{\frac{1}{8}}\left(\frac{5}{8}\right)^{\frac{3}{8}}}{(9)^{\frac{1}{4}}(9)^{\frac{1}{4}}(4)^{\frac{1}{8}}(1{,}000{,}000)^{\frac{3}{8}}}$$

$$=\frac{\left(\frac{1}{4}\right)^{\frac{3}{8}}\left(\frac{1}{8}\right)^{\frac{1}{8}}\left(\frac{5}{8}\right)^{\frac{3}{8}}}{(1{,}000{,}000)^{\frac{3}{8}}}=\$0.002\,. \qquad (54)$$

From (53) we get the production of the factory as,

$$M = \frac{3\left(\frac{1}{4}\right)^{\frac{1}{2}}\left(\frac{1}{8}\right)^{\frac{1}{8}}(1{,}000{,}000)^{\frac{5}{8}}}{(9)^{\frac{1}{4}}(9)^{\frac{1}{4}}(4)^{\frac{1}{8}}\left(\frac{5}{8}\right)^{\frac{5}{8}}}$$

$$= \frac{\left(\frac{1}{4}\right)^{\frac{1}{2}}\left(\frac{1}{8}\right)^{\frac{1}{8}}(1{,}000{,}000)^{\frac{5}{8}}}{2\times\left(\frac{5}{8}\right)^{\frac{5}{8}}} = 1{,}454 \text{ units.} \quad (55)$$

8. Analysis of Lagrange Multiplier for Production

As Lagrange multiplier is very important device for the prediction of future production, we want to provide a mathematical interpretation of it [30, 31]. By the chain rule of multivariate calculus we can take partial differentiation of M in equation (17) with respect to the budget B, and we can write,

$$\frac{\partial M}{\partial B} = M_K\frac{\partial K}{\partial B} + M_L\frac{\partial L}{\partial B} + M_R\frac{\partial R}{\partial B}. \quad (56)$$

Equation (56) is the marginal production of the factory. Now we use (56) to interpret the nature of Lagrange multiplier in the following Theorem 3:

Theorem 3: Prove that in optimization problems, Lagrange multiplier μ^* is positive and represents the shadow price.

Proof: From equation (31) taking the first-order derivatives we get,

$$M_K = xAK^{x-1}L^yR^z,$$
$$M_L = yAK^xL^{y-1}R^z, \quad (57)$$
$$M_R = zAK^xL^yR^{z-1}.$$

Using required values from (57) in (56) we yield,

$$\frac{\partial M}{\partial B} = xAK^{x-1}L^yR^z\frac{\partial K}{\partial B} + yAK^xL^{y-1}R^z\frac{\partial L}{\partial B} + zAK^xL^yR^{z-1}\frac{\partial R}{\partial B}. \quad (58)$$

From (32) we get,

$$xAK^{x-1}L^yR^z = r\mu,$$
$$yAK^xL^{y-1}R^z = w\mu, \quad (59)$$
$$zAK^xL^yR^{z-1} = \rho\mu.$$

Using (59) in (58) we get,

$$\frac{\partial M}{\partial B} = r\mu\frac{\partial K}{\partial B} + w\mu\frac{\partial L}{\partial B} + \rho\mu\frac{\partial R}{\partial B}$$

$$= \mu\left(r\frac{\partial K}{\partial B} + w\frac{\partial L}{\partial B} + \rho\frac{\partial R}{\partial B}\right). \quad (60)$$

Taking partial differentiation of (18) with respect to B, and by considering r, w, and ρ constants, we get,

$$1 = r\frac{\partial K}{\partial B} + w\frac{\partial L}{\partial B} + \rho\frac{\partial R}{\partial B}. \quad (61)$$

Comparing (60) and (61) we can write,

$$\frac{\partial M}{\partial B} = \mu$$

$$\Rightarrow \frac{\partial M^*}{\partial B} = \mu^*. \quad (62)$$

Therefore, (62) verifies (28). Here the Lagrange multiplier μ^* may be interpreted as the marginal production of the factory, i.e., if the factory wants to increase (decrease) 1 unit of its production, it would cause the total budget, B to increase (decrease) by approximately μ^* units. Therefore, Lagrange multiplier μ^* is the shadow price in the production procedures. Hence, the Lagrange multiplier provides a reasonable interpretation in production of a factory. Thus, the theorem is proved.

9. Verification of the Maximum Production

In this section we will try with the second-order partial differentiation with sufficient conditions of optimization. Of course, we will test whether the maximum production of a factory is possible or not, by using mathematical economics model [4]. Let us consider the determinant of the Hessian matrix,

$$|H| = \begin{vmatrix} 0 & -B_K & -B_L & -B_R \\ -B_K & V_{KK} & V_{KL} & V_{KR} \\ -B_L & V_{LK} & V_{LL} & V_{LR} \\ -B_R & V_{RK} & V_{RL} & V_{RR} \end{vmatrix}. \quad (63)$$

For maximum production, $|H| < 0$. Now we want to confirm that the production function that is obtained in equation (43) is really maximum [30, 31]. The following Theorem 4 clarifies the concept of maximum production.

Theorem 4: Prove that the production by the use of Cobb-

Douglas production function is indeed a maximum.

Proof: In equation (43), we have obtained the maximum production function. Now we will try to verify that it is really a maximum. Here, we use the results obtain in equations (18) and (32). Taking first-order partial differentiations of (18) we get,

$$B_K = r\ ,\ B_L = w\ , \text{and}\ B_R = \rho\ .$$

Taking second-order partial derivatives of (32) we get,

$$V_{KK} = x(x-1)AK^{x-2}L^y R^z\ ,$$

$$V_{LL} = y(y-1)AK^x L^{y-2} R^z\ ,$$

$$V_{RR} = z(z-1)AK^x L^y R^{z-2}\ .$$

Taking cross-order partial derivatives of (32) we get,

$$V_{KL} = V_{LK} = xyAK^{x-1}L^{y-1}R^z\ ,$$

$$V_{KR} = V_{RK} = xzAK^{x-1}L^y R^{z-1}\ , \tag{64}$$

$$V_{LR} = V_{RL} = yzAK^x L^{y-1} R^{z-1}\ .$$

Now we expand the Hessian (63) as,

$$|H| = B_K \begin{vmatrix} -B_K & V_{KL} & V_{KR} \\ -B_L & V_{LL} & V_{LR} \\ -B_R & V_{RL} & V_{RR} \end{vmatrix} - B_L \begin{vmatrix} -B_K & V_{KK} & V_{KR} \\ -B_L & V_{LK} & V_{LR} \\ -B_R & V_{RK} & V_{RR} \end{vmatrix} + B_R \begin{vmatrix} -B_K & V_{KK} & V_{KL} \\ -B_L & V_{LK} & V_{LL} \\ -B_R & V_{RK} & V_{RL} \end{vmatrix}$$

$$= B_K \left\{ -B_K \begin{vmatrix} V_{LL} & V_{LR} \\ V_{RL} & V_{RR} \end{vmatrix} - V_{KL} \begin{vmatrix} -B_L & V_{LR} \\ -B_R & V_{RR} \end{vmatrix} + V_{KR} \begin{vmatrix} -B_L & V_{LL} \\ -B_R & V_{RL} \end{vmatrix} \right\}$$
$$- B_L \left\{ -B_K \begin{vmatrix} V_{LK} & V_{LR} \\ V_{RK} & V_{RR} \end{vmatrix} - V_{KK} \begin{vmatrix} -B_L & V_{LR} \\ -B_R & V_{RR} \end{vmatrix} + V_{KR} \begin{vmatrix} -B_L & V_{LK} \\ -B_R & V_{RK} \end{vmatrix} \right\}$$
$$+ B_R \left\{ -B_K \begin{vmatrix} V_{LK} & V_{LL} \\ V_{RK} & V_{RL} \end{vmatrix} - V_{KK} \begin{vmatrix} -B_L & V_{LL} \\ -B_R & V_{RL} \end{vmatrix} + V_{KL} \begin{vmatrix} -B_L & V_{LK} \\ -B_R & V_{RK} \end{vmatrix} \right\}$$

$$= -B_K^2 V_{LL} V_{RR} + B_K^2 V_{LR}^2 + B_K B_L V_{KL} V_{RR} - B_K B_R V_{KL} V_{LR} - B_K B_L V_{KR} V_{LR} + B_K B_R V_{KR} V_{LL}$$
$$+ B_K B_L V_{KL} V_{RR} - B_K B_L V_{LR} V_{KR} - B_L^2 V_{KK} V_{RR} + B_L B_R V_{KK} V_{LR} + B_L^2 V_{KR}^2 - B_L B_R V_{KR} V_{KL}$$
$$- B_K B_R V_{KL} V_{LR} + B_K B_R V_{KR} V_{LL} + B_L B_R V_{KK} V_{LR} - B_R^2 V_{KK} V_{LL} - B_L B_R V_{KL} V_{KR} + B_R^2 V_{KL}^2$$

$$= -B_K^2 V_{LL} V_{RR} + B_K^2 V_{LR}^2 + 2B_K B_L V_{KL} V_{RR} - 2B_K B_R V_{KL} V_{LR} - 2B_K B_L V_{KR} V_{LR} + 2B_K B_R V_{KR} V_{LL}$$
$$- B_L^2 V_{KK} V_{RR} + 2B_L B_R V_{KK} V_{LR} + B_L^2 V_{KR}^2 - 2B_L B_R V_{KR} V_{KL} - B_R^2 V_{KK} V_{LL} + B_R^2 V_{KL}^2 . \tag{65}$$

Now putting the values from (64) in (65) we can write the Hessian as,

$$|H| = -y(y-1)z(z-1)A^2 r^2 K^{2x} L^{2y-2} R^{2z-2} + y^2 z^2 A^2 r^2 K^{2x} L^{2y-2} R^{2z-2} + 2xyz(z-1)A^2 rwK^{2x-1} L^{2y-1} R^{2z-2}$$
$$- 2xy^2 zA^2 r\rho K^{2x-1} L^{2y-2} R^{2z-1} - 2xyz^2 A^2 rwK^{2x-1} L^{2y-1} R^{2z-2} + 2xy(y-1)zA^2 r\rho K^{2x-1} L^{2y-2} R^{2z-1}$$
$$- x(x-1)z(z-1)A^2 w^2 K^{2x-2} L^{2y} R^{2z-2} + 2x(x-1)yzA^2 w\rho K^{2x-2} L^{2y-1} R^{2z-1} + x^2 z^2 A^2 w^2 K^{2x-2} L^{2y} R^{2z-2}$$
$$- 2x^2 yzA^2 w\rho K^{2x-2} L^{2y-1} R^{2z-1} - x(x-1)y(y-1)A^2 \rho^2 K^{2x-2} L^{2y-2} R^{2z} + x^2 y^2 A^2 \rho^2 K^{2x-2} L^{2y-2} R^{2z}$$

$$= A^2 K^{2x} L^{2y} R^{2z} \left\{ \begin{array}{l} \dfrac{-y(y-1)z(z-1)r^2}{L^2 R^2} + \dfrac{y^2 z^2 r^2}{L^2 R^2} + \dfrac{2xyz(z-1)rw}{KLR^2} - \dfrac{2xy^2 zr\rho}{KL^2 R} - \dfrac{2xyz^2 rw}{KLR^2} \\ + \dfrac{2xy(y-1)zr\rho}{KL^2 R} - \dfrac{x(x-1)z(z-1)w^2}{K^2 R^2} + \dfrac{2x(x-1)yzw\rho}{K^2 LR} + \dfrac{x^2 z^2 w^2}{K^2 R^2} \\ - \dfrac{2x^2 yzw\rho}{K^2 LR} - \dfrac{x(x-1)y(y-1)\rho^2}{K^2 L^2} + \dfrac{x^2 y^2 \rho^2}{K^2 L^2} \end{array} \right\}$$

$$= A^2 K^{2x} L^{2y} R^{2z} \left\{ \begin{array}{l} \dfrac{r^2 y^2 z}{L^2 R^2} + \dfrac{r^2 yz^2}{L^2 R^2} - \dfrac{r^2 yz}{L^2 R^2} - \dfrac{2rwxyz}{LKR^2} - \dfrac{2r\rho xyz}{KL^2 R} + \dfrac{w^2 x^2 z}{K^2 R^2} \\ + \dfrac{w^2 xz}{K^2 R^2} - \dfrac{w^2 xz}{K^2 R^2} - \dfrac{2w\rho xyz}{K^2 LR} + \dfrac{\rho^2 x^2 y}{K^2 L^2} + \dfrac{\rho^2 xy}{K^2 L^2} - \dfrac{\rho^2 xy}{K^2 L^2} \end{array} \right\} . \tag{66}$$

Now substituting the values of *K*, *L*, and *R* from (63) in equation (66) we get,

$$
\left|H\right| = A^2\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}
\left\{
\begin{aligned}
& r^2y^2z\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 + r^2yz^2\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 - r^2yz\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 \\
& -2rwxyz\left(\frac{r\Gamma}{xB}\right)\left(\frac{w\Gamma}{yB}\right)\left(\frac{\rho\Gamma}{zB}\right)^2 - 2r\rho xyz\left(\frac{r\Gamma}{xB}\right)\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right) \\
& +w^2x^2z\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 + w^2xz^2\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 - w^2xz\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 \\
& -2w\rho xyz\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{w\Gamma}{yB}\right)\left(\frac{\rho\Gamma}{zB}\right) + \rho^2x^2y\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{w\Gamma}{yB}\right)^2 \\
& +\rho^2xy^2\left(\frac{w\Gamma}{xB}\right)^2\left(\frac{w\Gamma}{yB}\right)^2 - \rho^2xy\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{w\Gamma}{yB}\right)^2
\end{aligned}
\right\}
$$

$$
= A^2\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\frac{r^2w^2\rho^2\Gamma^4}{B^4}\left(\frac{1}{z}+\frac{1}{y}-\frac{1}{yz}-\frac{2}{z}-\frac{2}{y}+\frac{1}{z}+\frac{1}{x}-\frac{1}{xz}-\frac{2}{x}+\frac{1}{y}+\frac{1}{x}-\frac{1}{xy}\right)
$$

$$
= -A^2\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\frac{r^2w^2\rho^2\Gamma^4}{B^4}\left(\frac{1}{yz}+\frac{1}{xz}+\frac{1}{xy}\right)
$$

$$
= -A^2\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\frac{r^2w^2\rho^2\Gamma^4}{B^4}\left[\frac{x+y+z}{xyz}\right]
$$

$$
= -A^2\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\frac{r^2w^2\rho^2\Gamma^4}{B^4}\times\frac{\Gamma}{xyz}
$$

$$
= -A^2\left(\frac{x^{2x}y^{2y}z^{2z}B^{2\Gamma}}{r^{2x}w^{2y}\rho^{2z}\Gamma^{2\Gamma}}\right)\left(\frac{r^2w^2\rho^2\Gamma^5}{xyzB^4}\right). \tag{67}
$$

Since $A>0$, $x, y, z>0$, and *r*, *w*, ρ are the rates of inputs *K*, *L*, and *R*, respectively and hence all are positive; while *B* is budget, which will never be negative, therefore, $\left|H\right|<0$, as required for maximum production. Thus, the value of the production function obtained in (43) is indeed a relative maximum value. Thus, the theorem is proved [15, 16, 31].

10. Prediction of Production by Economic Analysis

In this section we will try to predict the effects of the inputs for the maximum production. In our production procedure there are sixteen partial derivatives; $\frac{\partial K}{\partial r}$,…, $\frac{\partial L}{\partial r}$,…, $\frac{\partial R}{\partial r}$,…,$\frac{\partial \lambda}{\partial r}$, …, etc., which are referred as the comparative statics. In our analysis we will try some of them and will try to predict the production procedures by the changes in the costs of capital, labor, and other inputs with the economic analysis [30, 31].

As the second order condition is met, so the determinant of (63) does not vanish at the optimum, that is, $\left|J\right| = \left|H\right|$; consequently we apply the implicit function theorem. Let *F* be the vector-valued function of eight variables $\mu^*, K^*, L^*, R^*, r, w, \rho$, and *B*, and we define the function *F* for the point $\left(\mu^*, K^*, L^*, R^*, r, w, \rho, B\right)\in R^8$, and take the values in R^4. By the Implicit Function Theorem, the equation

$$
F\left(\mu^*, K^*, L^*, R^*, r, w, \rho, B\right) = 0\,, \tag{68}
$$

may be solved in the form of

$$
\begin{bmatrix} \mu^* \\ K^* \\ L^* \\ R^* \end{bmatrix} = G\left(r, w, \rho, B\right)\cdot \tag{69}
$$

Moreover, the Jacobian matrix for *G* is given by,

$$\begin{bmatrix} \frac{\partial\mu^*}{\partial r} & \frac{\partial\mu^*}{\partial w} & \frac{\partial\mu^*}{\partial\rho} & \frac{\partial\mu^*}{\partial B} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial\rho} & \frac{\partial K^*}{\partial B} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial\rho} & \frac{\partial L^*}{\partial B} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial\rho} & \frac{\partial R^*}{\partial B} \end{bmatrix} = -J^{-1}\begin{bmatrix} -K^* & -L^* & -R^* & 1 \\ -\mu^* & 0 & 0 & 0 \\ 0 & -\mu^* & 0 & 0 \\ 0 & 0 & -\mu^* & 0 \end{bmatrix}, \quad (70)$$

where the i[th] row in (70) on the right is obtained by differentiating the i[th] left side in (20a-d) with respect to r, then w, then ρ, and then B. Let C_{ij} be the cofactor of the element in the i[th] row and j[th] column of J, and then inverting J using the method of cofactor gives;

$J^{-1} = \frac{1}{|J|}C^T$, where $C = \left(C_{ij}\right)$. Hence (70) can be expressed as,

$$\begin{bmatrix} \frac{\partial\mu^*}{\partial r} & \frac{\partial\mu^*}{\partial w} & \frac{\partial\mu^*}{\partial\rho} & \frac{\partial\mu^*}{\partial B} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial\rho} & \frac{\partial K^*}{\partial B} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial\rho} & \frac{\partial L^*}{\partial B} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial\rho} & \frac{\partial R^*}{\partial B} \end{bmatrix} = -\frac{1}{|J|}C^T\begin{bmatrix} -K^* & -L^* & -R^* & 1 \\ -\mu^* & 0 & 0 & 0 \\ 0 & -\mu^* & 0 & 0 \\ 0 & 0 & -\mu^* & 0 \end{bmatrix},$$

$$= -\frac{1}{|J|}\begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix}\begin{bmatrix} -K^* & -L^* & -R^* & 1 \\ -\mu^* & 0 & 0 & 0 \\ 0 & -\mu^* & 0 & 0 \\ 0 & 0 & -\mu^* & 0 \end{bmatrix}$$

$$= -\frac{1}{|J|}\begin{bmatrix} -K^*C_{11}-\mu^*C_{21} & -L^*C_{11}-\mu^*C_{31} & -R^*C_{11}-\mu^*C_{41} & C_{11} \\ -K^*C_{12}-\mu^*C_{22} & -L^*C_{12}-\mu^*C_{32} & -R^*C_{12}-\mu^*C_{42} & C_{12} \\ -K^*C_{13}-\mu^*C_{23} & -L^*C_{13}-\mu^*C_{33} & -R^*C_{13}-\mu^*C_{43} & C_{13} \\ -K^*C_{14}-\mu^*C_{24} & -L^*C_{14}-\mu^*C_{34} & -R^*C_{14}-\mu^*C_{44} & C_{14} \end{bmatrix}. \quad (71)$$

In (71) total 16 comparative statics are available and we will try some of them in Theorems 5 to 7 to predict the economic analysis for the maximum production [4, 15, 16].

Theorem 5: Prove $\frac{\partial L^*}{\partial r} = 0$; there is no effect on the level of labor L, if the interest rate of capital K increases.

Proof: Now we examine the effects on labor L when the interest rate of capital K increases. From (71), we can write,

$$\frac{\partial L^*}{\partial r} = -\frac{1}{|J|}\left[-K^*C_{13} - \mu^*C_{23}\right]$$

$$= \frac{K^*}{|J|}\left[C_{13}\right] - \frac{\mu^*}{|J|}\left[C_{23}\right]$$

$$= \frac{K^*}{|J|} \times \text{Cofactor } C_{13} - \frac{\mu^*}{|J|} \times \text{Cofactor } C_{23}$$

$$= \frac{K^*}{|J|}\begin{vmatrix} -B_K & V_{KK} & V_{KR} \\ -B_L & V_{LK} & V_{LR} \\ -B_R & V_{RK} & V_{RR} \end{vmatrix} - \frac{\mu^*}{|J|}\begin{vmatrix} 0 & -B_K & -B_R \\ -B_L & V_{LK} & V_{LR} \\ -B_R & V_{RK} & V_{RR} \end{vmatrix}$$

$$= \frac{K^*}{|J|}\left\{-B_K\left(V_{LK}V_{RR} - V_{RK}V_{LR}\right) - V_{KK}\left(-B_LV_{RR} + B_RV_{LR}\right) + V_{KR}\left(-B_LV_{RK} + B_RV_{LK}\right)\right\}$$

$$-\frac{\mu^*}{|J|}\left\{B_K\left(-B_LV_{RR} + B_RV_{LR}\right) - B_R\left(-B_LV_{RK} + B_RV_{LK}\right)\right\}$$

$$\begin{aligned}\frac{\partial L^*}{\partial r} &= \frac{K^*}{|J|}\left\{-B_K V_{LK} V_{RR} + B_K V_{RK} V_{LR} + B_L V_{KK} V_{RR} - B_R V_{KK} V_{LR} - B_L V_{KR}^2 + B_R V_{KR} V_{LK}\right\} \\ &\quad - \frac{\mu^*}{|J|}\left\{-B_K B_L V_{RR} + B_K B_R V_{LR} + B_L B_R V_{RK} - B_R^2 V_{LK}\right\}.\end{aligned} \tag{72}$$

Now substituting the necessary values from (64) in (72) we get,

$$\frac{\partial L^*}{\partial r} = \frac{K^*}{|J|}\begin{Bmatrix} -rxyz(z-1)A^2K^{2x-1}L^{2y-1}R^{2z-2} + rxyz^2A^2K^{2x-1}L^{2y-1}R^{2z-2} \\ +wx(x-1)z(z-1)A^2K^{2x-2}L^{2y}R^{2z-2} - \rho x(x-1)yzA^2K^{2x}L^{2y-1}R^{2z-1} \\ -wx^2z^2A^2K^{2x-2}L^{2y}R^{2z-2} + \rho x^2yzA^2K^{2x-2}L^{2y-1}R^{2z-1} \end{Bmatrix}$$

$$-\frac{\mu^*}{|J|}\left\{-rwz(z-1)AK^xL^yR^{z-2} + r\rho yzAK^xL^{y-1}R^{z-1} + w\rho xzAK^{x-1}L^yR^{z-1} - \rho^2xyAK^{x-1}L^{y-1}R^z\right\}$$

$$= \frac{K^*}{|J|}\times xzA^2K^{2x}L^{2y}R^{2z}\left\{-\frac{ry(z-1)}{KLR^2} + \frac{ryz}{KLR^2} + \frac{w(x-1)(z-1)}{K^2R^2} - \frac{\rho(x-1)y}{K^2LR} - \frac{wxz}{K^2R^2} + \frac{\rho xy}{K^2LR}\right\}$$

$$-\frac{\mu^*}{|J|}\times AK^xL^yR^z\left\{-\frac{rwz(z-1)}{R^2} + \frac{r\rho yz}{LR} + \frac{w\rho xz}{KR} - \frac{\rho^2xy}{KL}\right\}$$

$$\begin{aligned}\frac{\partial L^*}{\partial r} &= \frac{K^*}{|J|}\times xzA^2K^{2x}L^{2y}R^{2z}\left\{\frac{ry}{KLR^2} - \frac{wx}{K^2R^2} - \frac{wz}{K^2R^2} + \frac{w}{K^2R^2} + \frac{\rho y}{K^2LR}\right\} \\ &\quad -\frac{\mu^*}{|J|}\times AK^xL^yR^z\left\{-\frac{rwz^2}{R^2} + \frac{rwz}{R^2} + \frac{r\rho yz}{LR} + \frac{w\rho xz}{KR} - \frac{\rho^2xy}{KL}\right\}.\end{aligned} \tag{73}$$

Now putting the values of K, L, and R from (38), (39) and (40) in (73) we get,

$$\frac{\partial L^*}{\partial r} = -\frac{A^2}{|J|}\times xz\left(\frac{xB}{r\Gamma}\right)\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\begin{Bmatrix} ry\left(\frac{r\Gamma}{xB}\right)\left(\frac{w\Gamma}{yB}\right)\left(\frac{\rho\Gamma}{zB}\right)^2 - wx\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 - wz\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 \\ +w\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 + \rho y\left(\frac{r\Gamma}{xB}\right)^2\left(\frac{w\Gamma}{yB}\right)\left(\frac{\rho\Gamma}{zB}\right) \end{Bmatrix}$$

$$-\frac{A^2}{|J|}\times\frac{x^xy^yz^z}{r^xw^y\rho^z}\left(\frac{B^{\Gamma-1}}{\Gamma^{\Gamma-1}}\right)\left(\frac{xB}{r\Gamma}\right)^x\left(\frac{yB}{w\Gamma}\right)^y\left(\frac{zB}{\rho\Gamma}\right)^z\begin{Bmatrix} -rwz^2\left(\frac{\rho\Gamma}{zB}\right)^2 + rwz\left(\frac{\rho\Gamma}{zB}\right)^2 + r\rho yz\left(\frac{w\Gamma}{yB}\right)\left(\frac{\rho\Gamma}{zB}\right) \\ +w\rho xz\left(\frac{r\Gamma}{xB}\right)\left(\frac{\rho\Gamma}{zB}\right) - \rho^2xy\left(\frac{r\Gamma}{xB}\right)\left(\frac{w\Gamma}{yB}\right) \end{Bmatrix}$$

$$\begin{aligned}&= \frac{A^2}{|J|}\times xz\left(\frac{xB}{r\Gamma}\right)\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\left(\frac{r^2w\rho^2\Gamma^4}{B^4}\right)\left(\frac{1}{xz^2} - \frac{1}{xz^2} - \frac{1}{x^2z} + \frac{1}{x^2z^2} + \frac{1}{x^2z}\right) \\ &\quad -\frac{A^2}{|J|}\times\frac{x^xy^yz^z}{r^xw^y\rho^z}\left(\frac{B^{\Gamma-1}}{\Gamma^{\Gamma-1}}\right)\left(\frac{xB}{r\Gamma}\right)^x\left(\frac{yB}{w\Gamma}\right)^y\left(\frac{zB}{\rho\Gamma}\right)^z\left(\frac{rw\rho^2\Gamma^2}{B^2}\right)\left(-1 + \frac{1}{z} + 1 + 1 - 1\right)\end{aligned}$$

$$\begin{aligned}&= \frac{A^2}{|J|}\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\left(\frac{rw\rho^2\Gamma^3}{B^3}\right)\left(\frac{1}{z}\right) \\ &\quad -\frac{A^2}{|J|}\times\frac{x^xy^yz^z}{r^xw^y\rho^z}\left(\frac{B^{\Gamma-1}}{\Gamma^{\Gamma-1}}\right)\left(\frac{xB}{r\Gamma}\right)^x\left(\frac{yB}{w\Gamma}\right)^y\left(\frac{zB}{\rho\Gamma}\right)^z\left(\frac{rw\rho^2\Gamma^2}{B^2}\right)\left(\frac{1}{z}\right)\end{aligned}$$

$$=\frac{A^2}{|J|}\left\{\left(\frac{1}{z}\right)\left(\frac{x^{2x}y^{2y}z^{2z}}{r^{2x}w^{2y}\rho^{2z}}\right)\left(\frac{B^{2\Gamma}}{\Gamma^{2\Gamma}}\right)\left(\frac{rw\rho^2\Gamma^3}{B^3}\right)-\left(\frac{1}{z}\right)\left(\frac{x^{2x}y^{2y}z^{2z}}{r^{2x}w^{2y}\rho^{2z}}\right)\left(\frac{B^{2\Gamma}}{\Gamma^{2\Gamma}}\right)\left(\frac{rw\rho^2\Gamma^3}{B^3}\right)\right\}$$

$$\Rightarrow \frac{\partial L^*}{\partial r}=0\,. \tag{74}$$

Hence, the theorem is proved.

From (74) we observe that there will be no effect on the level of labor L, if the interest rate of capital K increases, i.e., there is no relation between labor L and capital K in the production procedure of the factory. If r increases, the factory authority may or may not reduce its capital and even it may increase its capital if it needs more labors and wants to increase production. In this situation, the authority even may increase wage rate w, if necessary, for the maximum production. Similarly, the factory authority may decrease its capital if it does not want to increase its production, and it may not see the welfare of labors [15, 16, 24].

Theorem 6: Prove $\frac{\partial K^*}{\partial r}<0$; if the interest rate of the capital K increases, the factory may decrease the level of input capital K for its production.

Proof: Now we find out the effect on capital K when its interest rate, r increases. From the equation (71), we find that,

$$\frac{\partial K^*}{\partial r}=-\frac{1}{|J|}\left[-K^*C_{12}-\mu^*C_{22}\right]$$

$$=-\frac{K^*}{|J|}\left[C_{12}\right]+\frac{\mu^*}{|J|}\left[C_{22}\right]$$

$$=-\frac{K^*}{|J|}\times\text{Cofactor } C_{12}+\frac{\mu^*}{|J|}\times\text{Cofactor } C_{22}$$

$$=-\frac{K^*}{|J|}\begin{vmatrix}-B_K & V_{KL} & V_{KR}\\ -B_L & V_{LL} & V_{LR}\\ -B_R & V_{RL} & V_{RR}\end{vmatrix}+\frac{\mu^*}{|J|}\begin{vmatrix}0 & -B_L & -B_R\\ -B_L & V_{LL} & V_{LR}\\ -B_R & V_{RL} & V_{RR}\end{vmatrix}$$

$$=-\frac{K^*}{|J|}\left\{-B_K\left(V_{LL}V_{RR}-V_{LR}V_{LR}\right)-V_{KL}\left(-B_LV_{RR}+B_RV_{LR}\right)+V_{KR}\left(-B_LV_{LR}+B_RV_{LL}\right)\right\}$$
$$+\frac{\mu^*}{|J|}\left\{B_L\left(-B_LV_{RR}+B_RV_{LR}\right)-B_R\left(-B_LV_{LR}+B_RV_{LL}\right)\right\}$$

$$=-\frac{K^*}{|J|}\left\{-B_KV_{LL}V_{RR}+B_KV_{LR}V_{LR}+B_LV_{KL}V_{RR}-B_RV_{KL}V_{LR}-B_LV_{KR}V_{LR}+B_RV_{KR}V_{LL}\right\}$$
$$+\frac{\mu^*}{|J|}\left\{-B_LB_LV_{RR}+2B_LB_RV_{LR}-B_RB_RV_{LL}\right\}. \tag{75}$$

Using the values from (64) in (75) we get,

$$\frac{\partial K^*}{\partial r}=-\frac{K^*}{|J|}\begin{Bmatrix}-ry(y-1)z(z-1)A^2K^{2x}L^{2y-2}R^{2z-2}+ry^2z^2A^2K^{2x}L^{2y-2}R^{2z-2}\\ +wxyz(z-1)A^2K^{2x-1}L^{2y-1}R^{2z-2}-\rho xy^2zA^2K^{2x-1}L^{2y-2}R^{2z-1}\\ -wxyz^2A^2K^{2x-1}L^{2y-1}R^{2z-2}+\rho xy(y-1)zA^2K^{2x-1}L^{2y-2}R^{2z-1}\end{Bmatrix}$$
$$+\frac{\mu^*}{|J|}\left\{-w^2z(z-1)AK^xL^yR^{z-2}+2w\rho yzAK^xL^{y-1}R^{z-1}-\rho^2y(y-1)AK^xL^{y-2}R^z\right\}$$

$$=-\frac{K^*}{|J|}\left(yzA^2K^{2x}L^{2y}R^{2z}\right)\left\{-\frac{r(y-1)(z-1)}{L^2R^2}+\frac{ryz}{L^2R^2}+\frac{wx(z-1)}{KLR^2}-xy-\frac{wxz}{KLR^2}+\frac{\rho x(y-1)}{KL^2R}\right\}$$
$$+\frac{\mu^*}{|J|}\left(AK^xL^yR^z\right)\left\{-\frac{w^2z(z-1)}{R^2}+\frac{2w\rho yz}{LR}-\frac{\rho^2z(z-1)}{L^2}\right\}$$

$$=-\frac{K^*}{|J|}\left(yzA^2K^{2x}L^{2y}R^{2z}\right)\left\{\frac{ry}{L^2R^2}+\frac{rz}{L^2R^2}-\frac{r}{L^2R^2}-\frac{wx}{KLR^2}-\frac{\rho x}{KL^2R}\right\}$$
$$+\frac{\mu^*}{|J|}\left(AK^xL^yR^z\right)\left\{-\frac{w^2z^2}{R^2}+\frac{w^2cz}{R^2}+\frac{2w\rho yz}{LR}-\frac{\rho^2y^2}{L^2}+\frac{\rho^2y}{L^2}\right\}. \quad (76)$$

Now using the values of K, L, and R from (38), (39) and (40) in (76) we get,

$$\frac{\partial K^*}{\partial r}=-\frac{A^2}{|J|}\left(yz\left(\frac{xB}{r\Gamma}\right)\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\right)\begin{Bmatrix}ry\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2+rz\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2-r\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2\\-wx\left(\frac{r\Gamma}{xB}\right)\left(\frac{w\Gamma}{yB}\right)\left(\frac{\rho\Gamma}{zB}\right)^2-\rho x\left(\frac{r\Gamma}{xB}\right)\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)\end{Bmatrix}$$

$$+\frac{A^2}{|J|}\left(\frac{x^xy^yz^z}{r^xw^y\rho^z}\right)\left(\frac{B^{\Gamma-1}}{\Gamma^{\Gamma-1}}\right)\left(\frac{xB}{r\Gamma}\right)^x\left(\frac{yB}{w\Gamma}\right)^y\left(\frac{zB}{\rho\Gamma}\right)^z\begin{Bmatrix}-w^2z^2\left(\frac{\rho\Gamma}{zB}\right)^2+w^2z\left(\frac{\rho\Gamma}{zB}\right)^2+2w\rho yz\left(\frac{w\Gamma}{yB}\right)\left(\frac{\rho\Gamma}{zB}\right)\\-\rho^2y^2\left(\frac{w\Gamma}{yB}\right)^2+\rho^2y\left(\frac{w\Gamma}{yB}\right)^2\end{Bmatrix}$$

$$=-\frac{A^2}{|J|}\left(yz\left(\frac{xB}{r\Gamma}\right)\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\right)\left(\frac{w^2\rho^2\Gamma^4}{B^4}\right)\left\{\frac{ry}{y^2z^2}+\frac{rz}{y^2z^2}-\frac{r}{y^2z^2}-\frac{r}{yz^2}-\frac{r}{y^2z}\right\}$$
$$+\frac{A^2}{|J|}\left(\frac{x^xy^yz^z}{r^xw^y\rho^z}\right)\left(\frac{B^{\Gamma-1}}{\Gamma^{\Gamma-1}}\right)\left(\frac{xB}{r\Gamma}\right)^x\left(\frac{yB}{w\Gamma}\right)^y\left(\frac{zB}{\rho\Gamma}\right)^z\left(\frac{w^2\rho^2\Gamma^2}{B^2}\right)\left(-1+\frac{1}{z}+2-1+\frac{1}{y}\right)$$

$$=\frac{-A^2x}{|J|}\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\left(\frac{w^2\rho^2\Gamma^3}{B^3}\right)\left(-\frac{1}{yz}\right)$$
$$+\frac{A^2}{|J|}\left(\frac{x^xy^yz^z}{r^xw^y\rho^z}\right)\left(\frac{B^{\Gamma-1}}{\Gamma^{\Gamma-1}}\right)\left(\frac{xB}{r\Gamma}\right)^x\left(\frac{yB}{w\Gamma}\right)^y\left(\frac{zB}{\rho\Gamma}\right)^z\left(\frac{w^2\rho^2\Gamma^2}{B^2}\right)\left(\frac{y+z}{yz}\right)$$

$$=\frac{A^2}{|J|}\begin{Bmatrix}\left(\frac{x}{yz}\right)\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\left(\frac{w^2\rho^2\Gamma^3}{B^3}\right)\\+\left(\frac{x^xy^yz^z}{r^xw^y\rho^z}\right)\left(\frac{B^{\Gamma-1}}{\Gamma^{\Gamma-1}}\right)\left(\frac{xB}{r\Gamma}\right)^x\left(\frac{yB}{w\Gamma}\right)^y\left(\frac{zB}{\rho\Gamma}\right)^z\left(\frac{w^2\rho^2\Gamma^2}{B^2}\right)\left(\frac{y+z}{yz}\right)\end{Bmatrix}$$

$$=\frac{A^2}{|J|}\left\{\left(\frac{x}{yz}\right)\left(\frac{x^{2x}y^{2y}z^{2z}}{r^{2x}w^{2y}\rho^{2z}}\right)\left(\frac{B^{2\Gamma}}{\Gamma^{2\Gamma}}\right)\left(\frac{w^2\rho^2\Gamma^3}{B^3}\right)+\left(\frac{x^{2x}y^{2y}z^{2z}}{r^{2x}w^{2y}\rho^{2z}}\right)\left(\frac{B^{2\Gamma}}{\Gamma^{2\Gamma}}\right)\left(\frac{w^2\rho^2\Gamma^3}{B^3}\right)\left(\frac{y+z}{yz}\right)\right\}$$

$$=\frac{A^2}{|J|}\left(\frac{x^{2x}y^{2y}z^{2z}}{r^{2x}w^{2y}\rho^{2z}}\right)\left(\frac{B^{2\Gamma}}{\Gamma^{2\Gamma}}\right)\left(\frac{w^2\rho^2\Gamma^3}{B^3}\right)\left(\frac{x}{yz}+\frac{y+z}{yz}\right)$$

$$\frac{\partial K^*}{\partial r}=\frac{A^2}{|J|}\left(\frac{x^{2x}y^{2y}z^{2z}}{r^{2x}w^{2y}\rho^{2z}}\right)\left(\frac{B^{2\Gamma}}{\Gamma^{2\Gamma}}\right)\left(\frac{w^2\rho^2\Gamma^4}{B^3}\right)\left(\frac{1}{yz}\right). \quad (77)$$

Since $|J|=|H|$, hence by substituting the value of $|H|$ from (71) into the equation (77), we get,

$$\frac{\partial K^*}{\partial r}=A^2\left(\frac{x^{2x}y^{2y}z^{2z}}{r^{2x}w^{2y}\rho^{2z}}\right)\left(\frac{B^{2\Gamma}}{\Gamma^{2\Gamma}}\right)\left(\frac{w^2\rho^2\Gamma^4}{B^3}\right)\left(\frac{1}{yz}\right)\times\frac{1}{-A^2\left(\frac{x^{2x}y^{2y}z^{2z}B^{2\Gamma}}{r^{2x}w^{2y}\rho^{2z}\Gamma^{2\Gamma}}\right)\left(\frac{r^2w^2\rho^2\Gamma^5}{xyzB^4}\right)}$$

$$\frac{\partial K^*}{\partial r} = -\frac{xB}{r^2\Gamma}. \tag{78}$$

Since $x, y, z > 0$, $r > 0$, and B is the budget of the factory that can never be negative, and hence (78) provides,

$$\frac{\partial K^*}{\partial r} < 0. \tag{79}$$

Equation (79) indicates that if the interest rate of the capital K increases, the factory may decrease the level of input capital K for the sustainability of its production [30, 31].

Theorem 7: Prove $\frac{\partial K^*}{\partial B} > 0$; the budget size of the factory increases, the level of input of capital K must be increased for the maximization of production.

Proof: Now we analyze the effects of *K*, *L*, and *R* if there is a change in budget *B*. If the factory authority wants to increase its production, it must try to increase its existing budget. In such a situation we will try to find the effects of *K*, *L*, and *R*. From equation (61), we get,

$$\frac{\partial K^*}{\partial B} = -\frac{1}{|J|} \times \text{Cofactor } C_{12}$$

$$= \frac{1}{|J|}\begin{vmatrix} -B_K & V_{KL} & V_{KR} \\ -B_L & V_{LL} & V_{LR} \\ -B_R & V_{RL} & V_{RR} \end{vmatrix}$$

$$= \frac{1}{|J|}\left\{-B_K\begin{vmatrix} V_{LL} & V_{LR} \\ V_{RL} & V_{RR}\end{vmatrix} - V_{KL}\begin{vmatrix} -B_L & V_{LR} \\ -B_R & V_{RR}\end{vmatrix} + V_{KR}\begin{vmatrix} -B_L & V_{LL} \\ -B_R & V_{RL}\end{vmatrix}\right\}$$

$$= \frac{1}{|J|}\left\{-B_K\left(V_{LL}V_{RR} - V_{LR}V_{RL}\right) - V_{KL}\left(-B_L V_{RR} + B_R V_{LR}\right) + V_{KR}\left(-B_L V_{RL} + B_R V_{LL}\right)\right\}$$

$$= \frac{1}{|J|}\left\{-B_K Z_{LL} Z_{RR} + B_K Z_{LR}^2 + B_L Z_{KL} Z_{RR} - B_R Z_{KL} Z_{LR} - B_L Z_{KR} Z_{RL} + B_R Z_{KR} Z_{LL}\right\}. \tag{80}$$

Now using the required values from (6) in (80) we get,

$$\frac{\partial K^*}{\partial B} = \frac{1}{|J|}\begin{Bmatrix} -ry(y-1)z(z-1)A^2K^{2x}L^{2y-2}R^{2z-2} + ry^2z^2A^2K^{2x}L^{2y-2}R^{2z-2} + wxyz(z-1)A^2K^{2x-1}L^{2y-1}R^{2z-2} \\ -\rho xy^2zA^2K^{2x-1}L^{2y-2}R^{2z-1} - wxyz^2A^2K^{2x-1}L^{2y-1}R^{2z-2} + \rho xy(y-1)zA^2K^{2x-1}L^{2y-2}R^{2z-1} \end{Bmatrix}$$

$$= \frac{1}{|J|} \times yzA^2K^{2x}L^{2y}R^{2z}\left\{-\frac{r(y-1)(z-1)}{L^2R^2} + \frac{ryz}{L^2R^2} + \frac{wx(z-1)}{KLR^2} - \frac{\rho xy}{KL^2R} - \frac{wxz}{KLR^2} + \frac{\rho x(y-1)}{KL^2R}\right\}.$$

$$= \frac{1}{|J|} \times yzA^2K^{2x}L^{2y}R^{2z}\left\{\frac{ry}{L^2R^2} + \frac{rz}{L^2R^2} - \frac{r}{L^2R^2} - \frac{wx}{KLR^2} - \frac{\rho x}{KL^2R}\right\}. \tag{81}$$

Now using the values of *K*, *L*, and *R* from (38), (39), and (40) in (81) we yield,

$$\frac{\partial K^*}{\partial B} = \frac{A^2}{|J|} \times yz\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\begin{Bmatrix} ry\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 + rz\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 - r\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right)^2 \\ -wx\left(\frac{r\Gamma}{xB}\right)\left(\frac{w\Gamma}{yB}\right)\left(\frac{\rho\Gamma}{zB}\right)^2 - \rho x\left(\frac{r\Gamma}{xB}\right)\left(\frac{w\Gamma}{yB}\right)^2\left(\frac{\rho\Gamma}{zB}\right) \end{Bmatrix}$$

$$= \frac{A^2}{|J|} \times yz\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\left(\frac{rw^2\rho^2\Gamma^4}{B^4}\right)\left(\frac{1}{yz^2} + \frac{1}{y^2z} - \frac{1}{y^2z^2} - \frac{1}{yz^2} - \frac{1}{y^2z}\right)$$

$$= \frac{A^2}{|J|}\left(\frac{xB}{r\Gamma}\right)^{2x}\left(\frac{yB}{w\Gamma}\right)^{2y}\left(\frac{zB}{\rho\Gamma}\right)^{2z}\left(\frac{rw^2\rho^2\Gamma^4}{B^4}\right)\left(-\frac{1}{yz}\right). \tag{82}$$

Since $|J| = |H|$, then by using the value of $|H|$ from (67) in (82) we get,

$$\frac{\partial K^*}{\partial B} = -\frac{1}{A^2}\left(\frac{r^{2x}w^{2y}\rho^{2z}\Gamma^{2\Gamma}}{x^{2x}y^{2y}z^{2z}B^{2\Gamma}}\right)\left(\frac{xyzB^4}{r^2w^2\rho^2\Gamma^5}\right)A^2\left(\frac{x^{2x}y^{2y}z^{2z}B^{2\Gamma}}{r^{2x}w^{2y}\rho^{2z}\Gamma^{2\Gamma}}\right)\left(-\frac{r^2w^2\rho^2\Gamma^5}{xyzB^4}\right)\left(\frac{x}{r\Gamma}\right)$$

$$\Rightarrow \frac{\partial K^*}{\partial B} = \frac{x}{r\Gamma}. \tag{83}$$

Since, $x > 0$, $r > 0$, and $\Gamma > 0$, hence (83) provides,

$$\frac{\partial K^*}{\partial B} > 0. \tag{84}$$

Thus, the theorem is proved.

The inequality (84) indicates that when the budget size of the factory increases the level of input of capital K must be increased for increasing the production. Similarly, when the budget size of the factory increases, we can find the other inequalities for L and R as, $\frac{\partial L^*}{\partial B} > 0$, and $\frac{\partial R^*}{\partial B} > 0$, respectively. Finally, we can say that if the factory authority wants to increase its budge, it must increase all inputs, such as capital K, labor L, and other inputs R, for the maximum production [30, 31, 37].

11. Conclusions and Recommendations

In this study we have analyzed the economic analysis of production function. For the sustainability of the factory it is needed a maximum production that must give maximum profit. We have analyzed the Cobb-Douglas production function to determine the maximum production of a factory. We observe that the value of the Lagrange multiplier indicates shadow price. We have used both Jacobian and Hessian to verify the optimization and predict about the future sustainable production procedures of the factory.

If production of a factory will increase, more labors and use of all other inputs will be increased. As a result, employment opportunity will be created, and production of raw materials will be increased. Consequently, national and global economic development will be increased in the long-run.

Bangladesh has shifted from agro-economy to industrial economy. At present in Bangladesh the industrial development is unsustainable. Both money market and capital market are not strong to increase private investment. The industrial policy in Bangladesh should be sustainable industrialization. The factories of the country should follow sustainable strategy in production. Government of Bangladesh should take necessary actions to make sustainable environment to increase production in the factories.

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