International Journal of Economics and Business Administration
Vol. 7, No. 2, 2021, pp. 56-69
http://www.aiscience.org/journal/ijeba
ISSN: 2381-7356 (Print); ISSN: 2381-7364 (Online)

Bubbles, Asset Prices and Endogenous Economic Growth

Diana Loubaki*

ISG, Department of Economics, Marien NGouabi University, Brazzaville, Congo Republic

Abstract

Endogenous growth models particularity is to be non-testable since their fundamental goal is the equilibrium existence and its stability study. In contrast however, theoretical finance aim is to analyze volatility of asset prices in our article, caused by the bubbles existence provided by anticipations on future financial asset prices, thus is unstable by its nature. The idea is to conjugate stability and instability together in order to leave emerge a kind of saddle path where both growth and finance move at the same constant rate. Since the existing heterogeneity among the both fields, doesn't allow the provision of the reason for which financial crisis transmit in real economy create crashes and booms in the whole economic system. Indeed, using an OLG model where the equivalency between growth and finance based on bubbles and asset prices theories is highlighted and forms the complete theory, the methodology provided makes the theory yields a testability character since the both fields are endogeneized. Then, we find that, "global long-run growth" i.e the sum of asset prices and GNP growth rates is the lacking puzzle of the story because it makes growth stability and instable finance interaction, admits the unique empirical Pareto optimal equilibrium existence, thus the locus where the whole economic dynamics is stable over time.

Keywords

Global Long-run Growth, Bubbles, Asset Prices, Volatility, Unique Empirical Pareto Optimal Equilibrium, Equilibrium Stability

Received: January 12, 2021 / Accepted: February 18, 2021 / Published online: April 28, 2021

1. Introduction

This paper provides a novel theory in the economic literature which consists on making endogenous growth models econometrically testable in order to provide empirical verifications of the results found in the same analysis. Then, economic policy can be conducted by the social planner. Since, the equilibrium existence and stability of the system dynamics are provided by growth theory conjunction with bubbles and asset prices, then, the unique stable equilibrium when exist, is empirically verified in the same analysis. However, finance science is based on econometrical methods i.e statistics and probability and in contrast, economic growth theory is based on analysis, geometry and algebra, thus makes difficult the provision of a general equilibrium model including the both analyses together. That heterogeneity of finance and growth theory investigation methods, explain why growth theory economists, consider finance as an exogenous system such as an *ad-hoc* part of increasing returns of the GNP which measure growth. Thus, this article show-off that, finance and economic growth interaction exists and once highlighted, can decrease asset prices volatility, and allow for growth theory models testability. Moreover, since embodied knowledge or human capital is the by-product of education [1], it allow for technological change speed existence which varies with the country's economic dynamics level relied to innovations through R&D ([2]) conducted by researchers in universities adapted in good production sector by engineers for high-tech products quality. Therefore, education needs continuous

* Corresponding author
E-mail address: diana.loubaki@gmail.com

investment or capital demand increase which can be consider to be an asset which prices may be volatile like financial asset prices endow of bubbles causing high volatility. Since the uncertainty on future asset prices, increases investment risks, that can lead the economy to a crisis highlighted by crashes and booms if too high and creates depression in real economy i.e the GDP decrease and unemployment increase, thus needs control in order to ensure the economic viability over time. Moreover, since volatility comes from finance field and the long run growth is proved to be stable over time [3, 4], therefore the connection of the both fields yields a mechanism called, "global long-run growth" defined as the sum of asset prices growth rates of the joint fields i.e human and physical capitals. We show in this article that, the stability of the asset prices volatility highlighted by Var(asset prices) tends to 0 over time (lemma3) corresponds to the locus where the equilibrium is includes inside a closed ball i.e a compact set, thus, "global long-run growth" is stable and the theoretical model is empirically testable. The contributions of this article hold on several aspects which are *first,* finance and growth theory interaction through the bubbles' theory caused by asset prices volatility in the capital market, *second,* incentives to invest in human capital accumulation financed by the loan market or credit becomes an asset endowed of an uncertainty due to its price and the difficulty to get a loan. Since capital and credit markets investors face stochastic investments caused by uncertainty in asset prices because of volatility caused by prices anticipations markets and technological change high speed deserves human capital increase, thus becomes a random variable contrasting the standard growth theory in which finance is always an exogenous component i.e no more faces an *ad-hoc* decision here, due to the introduction of the Miao-Wang investment theory based on asset prices and bubbles inside growth theory analysis. The literature based on endogenous growth theory stipulates that, increasing returns come from incentives to invest in human capital accumulation [5-7], technological change [8, 9]) innovations based on R&D ([2]) as well as, the interaction between human capital and technological change [1]. The literature of finance used, is focused on asset prices and bubbles' studies on the basis of the Miao and Wang investment theory[1] [10] added with analysis ([11]), ([12], [13]), ([14]) and ([15]) which aim is mostly how to decrease or better control asset prices volatility caused by the bubbles movement. Consequently, the article uses the standard asset pricing equation for equity under risk neutrality in a discrete time environment where the stock price of an asset is the sum of its dividend and its future discounted value introduced inside a growth analysis. The solution of the asset price is given by the sum of the fundamental and the bubble components where the movement of the bubbles in standard literature is the discounted interest rate. As in [10], in this article, the bubbles are attached to productive firms where decisions related to incentives to invest in human capital mostly written in an *ad-hoc* way except here with unbounded growth as in the standard growth theory first models but follow a probability law, then growth is no more unbounded, since it is includes in a range as a compact set where the solution always exist and finite. The connection of finance and growth theories also allow for estimations and tests of the "global long-run growth", $g^*=g+g^p$ where the g is deterministic variable, thus constant and belongs to real economy expressed by, $g=g^k+g^h$ such that, g^i, $i=k,h$ are the respective growth rates of physical capital, g^k and human capital, g^h whereas, the other is stochastic i.e the asset prices growth rate, $g^p=g^{p(k)}+g^{p(h)}$ where $g^{p(j)}$, $j=p(k), p(h)$ are the respective growth rates of financial assets, $g^{p(k)}$ and human capital assets, $g^{p(h)}$. Moreover, the article presentation is done like following, section2 introduces the bubbles in the growth literature, section3 introduces both financial assets and bubbles in the economic growth model, thus extends the agents' utility function, then, the equilibrium provided by private agents i.e firms' profit maximization and households' utility function maximization, yield the respective wealth equilibrium of the both agents' kind i.e w^* which is per-capita income equilibrium and per-capita capital, k^*. The equilibrium stability is proved in the empirical validity of the theory investigation provided, since data used are stationary. We find that, since asset prices in quantity, k_t at time t are observable, then asset prices volatility are *first* include in a compact set, then the stable equilibrium exist. *Second,* tests and estimation can be conducted since the fundamental variables follow the probability laws done after have proved the equivalency between mathematics and statistics. But if the observed data are not stationary i.e when standard econometric conditions for regression study are not verified that necessitates more improved econometrical methods like cointegration method of Engle and Granger in order to ensure the stability of long run global growth provided through the asset prices series, then the endogenous growth tested models method provided by this article can work. Section 4 study focuses on g^k i.e the economic growth rate of physical capital and announces theorem1 where development is consider to be an escalator increasing function that admits the equilibrium existence i.e a locus where development stage of a given country is successful, but its improvements over time, depends on financial development efficiency i.e investment possibility, thus, explains the observed development level heterogeneity in GNP levels and rates of growth observed around the world across countries, then it is

1 The Miao-Wang, (2018) investment theory used, stipulates that, firms have a value function obtained through their profit maximization program using the Bellman equation. The optimality of the equilibrium necessity, yields social planner minimize asset prices volatility both in level and rate of growth to ensure economic growth stability since the both components are linked for the better economic system evolution dynamics, thus it may be found a way for their evolution to be a challenge for the economy of the whole.

also a source of the economic growth. Section5, extends the previous analysis and introduces human capital as an asset like financial assets, thus yields two things emerge (theorem1) which are, *first* the existence of g^h the economic growth rate focused on human capital and its price, $g^{p(h)}$ i.e human capital asset prices growth rate, then the general equilibrium model built is endowed of three different components which are, *first* global asset prices volatility existence, $p(k_t+h_t)$ or a sum of financial and human capital asset prices where $p(k_t+h_t)=p(k_t)+p(h_t)=p(H_t)$ i.e it is a linear function. *Second,* global financial growth rate, $g^p=g^{p(k)}+g^{p(h)}$ i.e the sum of the respective financial asset prices growth rate, $g^{p(k)}$ and of human asset growth rate, $g^{p(h)}$ where $g^{p(k)}=(g(p(k_{t+1}))-g(p(k_t)))/g(p(k_t))=g(p(k_{t+1}))$ and $g^{p(h)}=(g(p(h_{t+1}))-g(p(h_t)))/g(p(h_t))=g(p(h_{t+1}))$. *Thus, the whole yields* global long-run economic growth rate, $g^*=g^p+g$ i.e the sum of physical capital and human capital growth rates, where $g^k=(I_{t+1}-I_t)/I_t$ is respectively defined as the rate of the difference between investment accumulation, $I_{t+1}-I_t$ and its current level, I_t at time t ([10]) and the same thing for human capital i.e $g^h=(g(h_{t+1})-g(h_t)))/g(h_t)$. Then we find that, *first,* $(p(H_t), g^p,g)$ are linked and admits an equilibrium such that, the second variable, g^p is provided by the first, $p(H_t)$, where $g^p=g^{p(k)}+g^{p(h)}$ is the equilibrium looked for. In contrast, $p(H_t)$ as a disorders' source nature. The third variable, $g=g^k+g^h$ is the consequence of g^p seen as an investment condition, thus their link expressed as a sum, $g^*=g^p+g$ is global long-run growth exist and also is growth of the global economy. The difficulty of this novel view holds on the measure to use for "global long-run growth" evaluation, g^* since g is measured by GNP or the GDP whereas, g^p measure is essentially based on econometrical tools. Therefore, the unification must hold on the two concepts which are in theory i.e the formulation and the measure, thus yields to empirical equilibrium estimable and testable existence since it is possible to establish an equivalency between mathematics (algebra, analysis) and statistics (econometrical analysis) which also render growth theory expressed as an econometrical function ready to be submit to regressions, tests and forecast.

2. The Model

In an overlapping generation world, the agent lives two periods the young and the old ages, consumes and trades firms' assets, i where $i\epsilon\{1,2,\ldots,n\}$ without any trading frictions and receives a dividend, d_t^i for the asset i. Then, the first period aggregate budget constraint of the representative agent can be written, $w_t+d_t\psi_t=c_t+p_t\psi_t$ where p_t is the vector of the asset prices at t such that, $p_t=(p_t^i)_i$ and $d_t=(d_t^i)_i$ is the dividend vector of the n *associate financial* assets at t, where $\psi_t=(\psi_t^i)_i$ is the quantity vector of the n assets hold by the representative agent from the representative firm, for $1\leq i\leq n$, $\Sigma_{i=1}^n\psi_{ti}=1$ and $R_t=1+r_t$ is the subjective interest rate. Thus, total dividend received by the representative household, d_t is such that, $d_t=\sum_{i=1}^n d_t^i$. In the second period, the representative agent consumes the fruit of his investment while in rest. Indeed, the second period budget constraint can be written such that, $c_{t+1}/1+r_t=d_{t+1}\psi_{t+1}/1+r_t$. Therefore, the intertemporal budget constraint of the representative agent can be written such that equation (1) i.e

$$c_t+p_t\psi_t+c_{t+1}/1+r_t=w_t+d_t\psi_t+d_{t+1}\psi_{t+1}/1+r_t \quad (1)$$

Lemma 1: *since each given financial asset, i of quantity,* k_t^i *admits a bubble component,* b_t^i *then, the average aggregate per-capita bubble,* $b_t=n^{-1}\Sigma_{i=1}^n b_t^i$ *exist for a given portfolio, expressed by equation* (2) i.e

$$b_t=b_{t+1}(1+\rho(q_t-1))/1+r_t \quad (2)$$

Where $q_t>1$ is the price of capital and $\rho>0$ is the poison probability for the firm to meet an investment opportunity. Equation (2) is provided by [10]

Proof: since the basic asset pricing equation for equity is, $p_t=d_t+p_{t+1}/1+r_t$ then, the solution is endowed of a bubble, $p_t=p_t^*+b_t$ where, p_t^* is the fundamental component, b_t is the bubble component such that, $b_t=\Sigma_{i=1}^n b_t^i$ which evolution rate equals the interest rate in the standard bubble theory i.e $b_{t+1}/b_t=1+r_t$, also known to be the condition of non arbitrage between bubbles and other assets. Following [10], per-capita bubbles raises investment by, $b_{t+1}/1+r_t$, and total discounted benefit of the bubbles, $(\rho(q_t-1)+1)b_{t+1}/1+r_t$, generates additional dividends, q_t-1 with a poison probability for the firm to meet an investment opportunity, $\rho\epsilon]0,1[$ where, $\rho(q_t-1)$ is the liquidity premium of capital. Indeed, equating the benefit with the cost of the bubbles, yields, $b_t=b_{t+1}(1+\rho(q_t-1))/1+r_t$ where $q_t>1$ is the price of capital.

2.1. The Household's Investment Strategy

Assumption 1: in financial market, firms sell financial assets that are bought by households and also borrow money to the banks in order to invest more in production.

From lemma1, since per-capita bubbles evolution rate is such that, $b_{t+1}/b_t<1+r_t$, then the asset pricing equation is expressed by (3) i.e

$$p_t=p_t^*+b_{t+1}(1+\rho(q_t-1))/1+r_t \quad (3)$$

Introducing (3) inside the intertemporal budget constraint of the agent, (1) yields, the budget constraint with assets and bubbles expressed by equation (4), i.e

$$w_t=c_t+c_{t+1}/1+r_t+d_{t+1}(\psi_t-\psi_{t+1})/1+r_t+b_t\psi_t \quad (4)$$

The utility function includes financial assets' demand, expressed by equation (5) i.e:

$$U(c_t,c_{t+1},\psi_t)=ln(c_t)+\beta ln(c_{t+1})+\mu ln(\psi_t)+\gamma ln(\psi_{t+1}) \quad (5)$$

Where, β, μ, $\gamma>0$ are the respective elasticities of the second period consumption, c_{t+1}, the first and the second period financial assets' hold in quantities, ψ_t and ψ_{t+1}. Solving the consumer's optimization program, yields, the agent's intertemporal wealth equilibrium, expressed by equation (6) i.e

$$w_t*=(1+\beta)c_t+d_{t+1}\theta/1+r_t \quad (6)$$

Where $\theta=\psi_t-\gamma/(\lambda b_t-\mu/\psi_t)$, $\psi_t\neq 0$ and $\lambda>0$ is the Lagrange multiplier or consumption goods price. Since wealth is spent both on consumption goods and on financial assets which brings dividends, thus is a positive variable in the wealth function and in contrast, the bubbles are negatively links to financial assets then decrease the agent's intertemporal wealth.

2.2. The Firm's Investment Strategy

The representative firm may live an indefinite time depending on its investment strategy and profit, in contrast, the representative agent lives only two periods. A given asset may live an indefinite time because of the inheritance mechanism due to altruism i.e the owner leaves it's financial assets to his descendent when he died, where an agent can have only one child. More precisely, in this part, the firm is able to finance its activities with its internal fund i.e its initial capital stock, $K_0>0$ which depreciates at a rate, $\delta>0$. The firm combines labor with a technology augmenting, $A(K_t)$ freely accessible (Romer, 1990) in order to produce output according to the Cobb-Douglas production function expressed by equation, (7) i.e

$$Y_t=(K_t)^{\alpha}(A(K_t)L_t)^{1-\alpha} \quad (7)$$

Where $\alpha\epsilon(0,1)$ is physical capital stock parameter. Profit maximization yields the equilibrium in wage rate income, (8) and in interest rate, (9) i.e

$$w_t=(1-\alpha)A(K_t)k_t^{\alpha} \quad (8)$$

$$R_t=1+r_t=\alpha k_t^{\alpha-1} \quad (9)$$

Where, $k_t=K_t/A(K_t)L_t$ is per-capita physical capital per efficiency unit expression. After production done, the representative firm needs to invest in new capital goods since it depreciates through the time where investment consists on transforming consumption goods one by one in capital per-efficiency units, k_t which accumulates through, k_{t+1} that the firm uses again in order to pursue operating. Following the economic growth literature, investment is expressed by, $I_t=k_{t+1}-(1-\delta)k_t$ where $\delta>0$ is the depreciation rate of per-capita physical capital level. Indeed, profit maximization provides the equilibrium, expressed by equations (8) and (9) which when include in the utility optimization of the household, where, $\alpha=1/2$, $A(K_t)=A$ yields the firm's equilibrium wealth, equation (10) i.e

$$k_t^*=c_t((1-\alpha)A-2(c_{t+1}-d_{t+1}\theta))^{-1} \quad (10)$$

Assumption 2: a given asset i at time t is endowed of a bubble, $b_t^i \approx N(0, \sigma^2)$, $i\epsilon\{1,2,\ldots,n\}$ where, σ^2 is the bubble volatility component

Lemma 2 *the average aggregate bubbles,* $b_t=n^{-1}\Sigma_{i=1}{}^n b_t^i$ *is stable since it follows* $N(0,\sigma^2/n)$ *according to assumption2. Otherwise, if* $b_t\rightarrow\infty$ *then it explodes, or if* $b_t\rightarrow 0$ *then it collapses.*

Proof: since a given financial assets *i* is endowed of a bubble, b_t^i (according to assumption2), then the sequences, $\{(b_t^i)_{1\leq i\leq n}\}\epsilon(I,d)$ exist and since it is defined on a metric space, I *belongs to* R^n where d is the associated distance, then a given portfolio of n assets, admits an average aggregate bubble component, $b_t=n^{-1}\Sigma_{i=1}{}^n b_t^i$ since by assumption2, $E(b_t^i)=0$ and $Var(b_t^i)=\sigma^2$ for $i\epsilon\{1,2,\ldots,n\}$, then, by the large number law, b_t converge to $N(0,\sigma^2/n)$. Moreover, b_t can also be written in algebra such that, $b_t=\sum_{i=1}{}^n x_i b_t^i$ since it admits a basis, $x=(x_1,x_2,\ldots,x_n)$ of b_t ϵI *belongs to* R^n, where I is also a vectorial space as a subset of R^n. Indeed, if $\sum_{i=1}{}^n x_i=1/n\rightarrow 0$ when $n\rightarrow\infty$ it yields, $x_i=0$ for all $i\epsilon\{1,2,\ldots,n\}$, then, $\{(b_t^i)_{1\leq i\leq n}\}$ are free, i.e $(b_t^i)_{1\leq i\leq n}$ are independent and identically distributed statistically, by the law $N(0,\sigma^2)$ thus, by Bienaymé-Tchebitchev theorem, per-capita aggregate bubbles, $b_t=\sum_{i=1}{}^n b_t^i/n$ converge in probability. Since, $E(b_t)=E(\sum_{i=1}{}^n b_t^i)/n=0$ and $Var(b_t)=Var(\sum_{i=1}{}^n b_t^i/n)=n\sigma^2/n^2=\sigma^2/n$ and by Bienayme-Tchebitchev theorem, it yields, $P(|b_t-b_0|< \;)\leq\sigma^2/n\rightarrow 0$ when $n\rightarrow\infty$ therefore, *first* b_t converge in probability to b_0 and b_t also converge in law to $\approx N(0,\sigma^2/n)$. Therefore, $(b_t^i)_{1\leq i\leq n}$ is a Cauchy sequences where each component converges to the same finite limit. Thus, b_t converge inside the closed ball of center 0 and of radius, $\sigma/n^{1/2}$ i.e $B(0,\sigma/n^{1/2})$, thus, is locally stable around the neighborhood, $[-\sigma/n^{1/2},+\sigma/n^{1/2}]$ which is a compact set i.e closed and bounded since $\sigma/n^{1/2}\epsilon]0,1[$ then, b_t admits a probability distribution function, $F(b_t)=1/\sigma(2\pi)^{1/2}exp\{-nb_t/2\sigma\}$, therefore b_t can be displayed on the plane such that if we adopted the distance, $d_2(b_t^i, b_t^j)=(\sum_{1\leq i,j\leq n}|b_t^i-b_t^j|)^{1/2}$, $i\neq j$ it yields, b_t *belongs to* $[-t_{1-\alpha/2}\sigma/(n)^{1/2};t_{1-\alpha/2}\sigma/(n)^{1/2}]\approx B(0, \; t_{1-\alpha/2}\sigma/(n)^{1/2})$, where $\sigma/n\epsilon]0,1[$ thus b_t converge to a finite limit when $n\rightarrow\infty$, indeed, stability is ensured. Otherwise, *if* b_t is such that, the volatility tends toward *0* for a given time, then the bubble, b_t collapses, thus yields the economic system to a crash. Otherwise, if the average aggregate bubble volatility, $\sigma^2/n>1$, then for a fixed n the bubbles, b_t explodes i.e goes toward an indeterminacy locus. Consequently, the young can't buy the old assets since their prices are too high. Therefore, what about the bubbles' volatility impact on asset prices?

2.3. Asset Prices Volatility

Since the firm value is measured by its financial asset prices vector, $p(k_t)=y_t$ *belongs to* R^{n+1} i.e $y_t=q_tk_t+b_t$ [2] where b_t is the average aggregate bubbles and q_t is capital cost vector (Miao-Wang, 2018), thus per-capita financial asset prices, $p(k_t)=y_t$ can also be written such that, $y_t=\sum_{i=0}^{n}(k_t^i\beta^i+\mathcal{E}_t^i)=\sum_{i=0}^{n}k_t^i\beta^i+\sum_{i=0}^{n}\mathcal{E}_t^i=k_t\beta+\mathcal{E}_t$ where $q_t=\beta$ is the parameter to estimate and b_t is the white noise expressed by, $\mathcal{E}_t$ thus the asset prices can be studied empirically, since it can be expressed such that, equation (11) i.e

$$y_t=k_t\beta+\mathcal{E}_t \tag{11}$$

Where k_t is financial assets demand of the representative agent's portfolio, $\beta=q_t=\{q_i\}_{1\leq i\leq n}$ is capital cost vector, $b_t=\mathcal{E}_t=\{\mathcal{E}_t^i\}_{1\leq i\leq n}$ is the bubble assimilated to a perturbation variable like a white noise thus, $\mathcal{E}_t$ follow $N(0,\sigma^2/n)$ $i\epsilon\{1,2,\ldots,n\}$. Moreover, since, $k_t=(k_t^i)_{1\leq i\leq n}$ are observations of n financial assets quantities, then $({}^tk_tk_t)$ is an inversible matrix i.e there exist $(k_t)^{-1}$ such that, $(k_t)^{-1}k_t=I$, thus can be diagonalized in order to prove its stability through the eigenvalues existence provided by the characteristic polynomial solutions, such that, if the trace or the eigenvalues sum is positive, in contrast to their product which should be negative, then, the economic system stability is ensured. Since the theory is verified because of the assumptions provided, then we have, $E(\mathcal{E}_t^i/k_t^1,\ k_t^2,\ldots,\ k_t^n)=0$ i.e asset prices and the bubbles are not correlated, $Var(\mathcal{E}_t)=\sigma^2/n$ and $Cov(\mathcal{E}_t^i,\mathcal{E}_t^j)=0,\ i\neq j$ where $i,j\ \epsilon\{1,2,\ldots,n\}$ meaning that, residuals are not correlated, the assets contain in the portfolio are not correlated among them too. In the following, figure 1, the red solid line, corresponds to asset prices convergence inside a closed ball since volatility belongs to a compact set, the yellow solid line corresponds to the bubbles' collapse since it tends to 0, thus yield the economic crash. Finally, the green solid line corresponds to the bubbles' explosion, thus to high asset prices volatility and panic in financial market. Where $Cov(k_t^i,k_t^j)=0$ for $i\neq j$ i.e exogenous variables are not correlated among them.

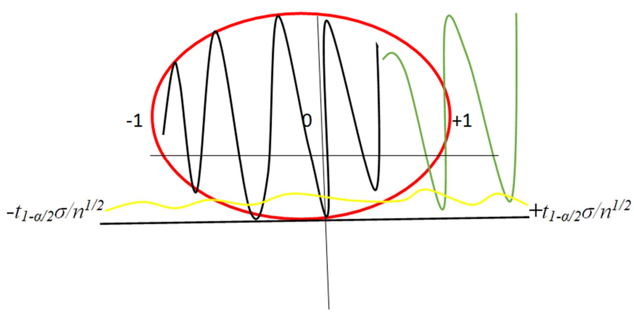

Figure 1. Asset Prices Dynamics.

The equivalency between algebra and topology, yields a basis existence, $(e_i)_{1\leq i\leq n}\ \epsilon I$ such that, each $\mathcal{E}_t\ \epsilon R^n$ implies, $\mathcal{E}_t=\sum_{i=1}^{n}\mathcal{E}_t^ie_i$, such that, $\sum_{i=1}^{n}e_i=0$ yields, $e_i=0$ for all $1\leq i\leq n$ i.e they are free, then $(\mathcal{E}_t^i)_{1\leq i\leq n}$ are not correlated among them and the same thing holds for $k_t=(k_t^i)_{1\leq i\leq n}$. Indeed, $Var(\mathcal{E}_t/k_t^1,\ k_t^2,\ldots,\ k_t^n)=\sigma^2I$ and $Var(\mathcal{E}_t)=\sigma^2I$.

Lemma 3: *the testable equilibrium in asset prices, $p(k^*)$ exist and is stable*

Proof: the previous model, expressed by equation (11) is MCO estimable i.e it can be determinate, β^* the estimator of β through the resolution of the program $Min\{(y_t-k_t\beta)^2\}$ where the first order condition, $\partial(y_t-k_t{}^t\beta)^2/\partial\beta=0$ yields, $2k_t{}^t(y_t-k_t{}^t\beta)=0$. Since, $k_ty_t=(k_t{}^tk_t)\beta$, then, capital cost estimation parameter is given by equation (12) i.e

$$\beta^*=({}^tk_tk_t)^{-1}(k_t{}^ty_t) \tag{12}$$

Since, equation (12) yields, $E(\beta^*)=\beta$ and $Var(\beta^*)=(k_t{}^tk_t)^{-1}\sigma^2$ thus, $\beta^*\rightarrow N({}^t\beta,(k_t{}^tk_t)^{-1}\sigma^2)$, indeed, the estimator β^* converge in law toward a normal distribution. Therefore, $(\beta\text{-}\beta^*)/(k_t{}^tk_t)^{-1/2}\sigma\approx N(0,1)$, allowing tests conduction. Thus, at the threshold, $1\text{-}t_\beta=95\%$, the parameter, β is such that, β *belongs to* $[-t_\beta(k_t{}^tk_t)^{-1/2}\sigma+\beta^*,+t_\beta(k_t{}^tk_t)^{-1/2}\sigma+\beta^*]$, i.e includes in a compact set, thus closed and bounded, therefore, converge to a finite limit, meaning that, *first,* the equilibrium in asset prices exist, $p(k^*)$, and is stable. *Second,* $p(k^*)$ is estimable and testable.

Assumption3: each financial assets' portfolio, $k_t=(k_t^1,k_t^2,\ \ldots,k_t^n)$, admits a corresponding financial asset prices vector, $p(k_t)=(p(k_t^1),\ p(k_t^2),\ldots,\ p(k_t^n))$ where, $k_t\approx N(m_k,\sigma_k)$, which volatility level, σ_k yields multiple equilibria such that, if $\sigma_k\ \epsilon\ ]0,1[$ *then,* k_t is stable, otherwise, if $\sigma_k\geq 1$, then k_t is non stable, thus, asset prices volatility, σ_k impact on k_t depends on $p(k_t)$ volatility over time i.e $Lim\{Var(p(k_t)\}$ *when* $t\rightarrow\infty$

Proposition1: *asset prices analytical function, $y_t=p(k_t)$ admits multiple estimable and testable equilibria*

Proof: financial asset stock prices function, $y_t=k_t\beta+\mathcal{E}_t$ is *first* an econometrical function, defined in algebra inside a metric space $(X,\ d)$ and takes it values on other metric space, (X',d') where $X,\ X'$ *belong to* R^{n+}. Thus, each vector of n given assets demand, $k_t=(k_t^i)_{1\leq i\leq n}\ \epsilon X$ *belongs to* R^{n+} admits at least one price vector, $p(k_t)=p(k_t^i)_{1\leq i\leq n}$ since each $\delta_{k0}>0$ and $k_0>0$ induce $\varepsilon_{k0}>0$ existence such that, $d(k_t^i,k_0)\leq\delta_{k0}$ yields $d'(p(k_t^i),l_{k0})\leq\varepsilon_{k0}$ then, the sequences, $\{(k_t^i),\ i\epsilon\{1,2,\ldots,n\}\}$ converge to k^* i.e $k_t=(k_t^i)_{1\leq i\leq n}\rightarrow k^*$ indeed, the asset prices sequences $p(k_t)=\{p(k_t^1),p(k_t^2),\ldots,p(k_t^n)\}\rightarrow p(k^*)\epsilon X'$ where $p(k^*)$ is a unique limit. Indeed, $(k^*,\ y^*)$ equilibrium exist and is stable. Since, the function of asset prices quantities, $p(k_t)$ is defined on a metric space, p is continuous for all given assets' vector, $(k_t^i)_{1\leq i\leq n}$, thus if the stable equilibrium, (k^*,y^*) is also the optimum, i.e $k^*=max\{k_t^i\}_{1\leq i\leq n}$ then $p(k^*)=Min\{p(k_t^i),i\epsilon(1,2,\ldots,n)\}\epsilon X'$ *belongs to* R^n. Therefore,

2 This formula is provided by Miao-Wang (2018)

$k_t=(k_t^i)_{1\leq i\leq n}\rightarrow k^*$ *belongs to* X its dynamics follow, $k_{t+1}=p(k_t)$ such that, $k_{t+1}\rightarrow p(k^*)=k^*$ is thus unique i.e stable. Indeed, *first*, $\{p(k_t^i),\ i\epsilon(1,2,\ldots,n)\}\rightarrow p(k^*)$ i.e $\{k_{t+1}{}^i=p(k_t^i)\}_{1\leq i\leq n}$ is a Cauchy sequence. *Second*, $k^*=p(k^*)$ a fixed point which yields the existence of an interior stable solution, thus, the assets, i in quantity, k_t^i for $1\leq i\leq n$ are not risky assets since their return rate, $R(k)=(R_1^k,R_2^k,\ldots,R_n^k)\rightarrow R(k^*)$ is constant i.e converge to a stable equilibrium, $R(k^*)$. Thus, households are interested by those financial assets that return or dividend, respectively $E(R(k))$ and $d_{t+1}/1+r_t$ are high, in contrast, $Var(R(k))$ or $Var(p(k))$ are low. Moreover, since risk measured by, $Cov(k_t^i,k_t^j\ )_{i\neq j}\ \rightarrow 0$, then financial assets, $i\epsilon(1,2,\ldots,n)\}$ of quantities, $(k_t^i)_{1\leq i\leq n}$ are independent, identically distributed by a normal distribution and without risk, thus $(k_t\text{-}E(k_t))/\sigma(k_t)\approx N(0,1)$ i.e k_t *belongs to* $B(0,\ t_\alpha\sigma(k_t)+E(k_t))$ where, $B=[-t_\alpha\sigma(k_t)+E(k_t);+t_\alpha\sigma(k_t)+E(k_t)]$. However, from lemma2, since $E(y_t)=E(k_t\beta+\mathcal{E}_t)=\beta m_k=m_y$ i.e asset prices expectation *and* $Var(y_t)=Var(k_t\beta+\mathcal{E}_t)=(\beta\sigma_k)^2+\sigma^2=\sigma_y^2$ then $y_t\ \approx N(m_y;\ \sigma_y^2\ )$, therefore, $y_t=p(k_t)$ *belongs to* $[-t_\alpha\sigma_y+m_y;+t_\alpha\sigma_y+m_y]=B(0,\ t_\alpha\sigma_y+m_y)$ *belongs to* $B(0,1)$ Otherwise, if $\sigma_y^2\ \rightarrow\infty$ i.e is unbounded, then $p(k)\rightarrow\infty$, consequently, *first* if $p(k_t)$ no more converge to $N(m_y;\ \sigma_y^2\ )$, thus is unbounded and admits an exterior unstable solution, it is no more possible to reach possible to have, $p(k_t)\text{-}E(p(k_t\ ))/\sigma_y\rightarrow N(0,1)$ *as* $t\rightarrow\infty$ meaning that, $p(k_t)$ *is* no more include inside the closed ball $[-t_\alpha\sigma_y+m_y,;+t_\alpha\sigma_y+m_y]=B(0,m_y+\sigma_y q_\alpha)$, the equilibrium, $p(k^*)$ can't be find Finally, if asset prices volatility, $\sigma^2{}_y\rightarrow 0$, then assets, $i\epsilon\{1,2,\ldots,n\}$ are not risky, thus gain lost probability equals 0 and the dividend is the same as before with inflation inclusion because risk, $Cov(k_t^i,k_t^j)_{i\neq j}\rightarrow 0,\ i,j\epsilon\{1,2,\ldots,n\}$, then the portfolio of i assets $i\epsilon\{1,2,\ldots,n\}$ is efficient, thus drives the investor to gain. For the risk adverse households, since $E(R(k))=R(k^*)$ is constant, then, the volatility of the return of the investment in financial asset, $Var(R(k^*))\approx 0$. Since, $\beta=1$ and $V(p(k_t))\epsilon]0,1[$, then, $p(k_t)$ converge to a finite limit, $p(k_t)$ is bounded due to not significant assets' prices anticipations, in contrast, if $d(Var(p(k_t^i),Var(p(k_t^j))>\alpha_{k,\varepsilon}>0\ \ i\neq j$ then, $d'(E(p(k_t^1),\ E(p(k_t^2)>\varepsilon_k$ meaning that, high volatility yields, high prices expectations, thus the equilibrium is difficult to establish because it may not exist, since asset prices distribution can't converge in law toward a given known probability distribution, the continuity of the function $p(k_t)$ is not ensured and it is no more possible to estimate its future value $p(k_{t+1})=E(p(k_t))$ since there doesn't exist $M>0$ such that, the function p is from Lipschitz. Therefore, $Var(p(k_t))$ converge to $+\infty$ necessarily, then the expectation of future asset prices, even such that, $m_k>1$, yields uncertainty in gain and dividends. Finally, if the expectation of the future asset prices, $E(p(k_t))=m_k\geq 1$ and $0<\beta<1$, then $E(p(k_t))$ converge to ε_0 where $Var(p(k_t))=\beta\sigma_k\rightarrow 0$, then $p(k_t)$ converge in law toward $N(\varepsilon_0,\sigma_{p(k)}{}^2)$, indeed, $p(k_t)$ *belongs to* $[-t_\alpha\sigma_y+\varepsilon_0;\ t_\alpha\sigma_y+\varepsilon_0]$ *belongs to* $B(0,\ t_\alpha\sigma_y+\varepsilon_0)$, because Y_a such that, $Y_a=(p(k_t)\text{-}\varepsilon_0)/\sigma_y\approx N(0,1)$ exist and is includes inside a compact set, thus closed and bounded, at the threshold, $1\text{-}\alpha$. Therefore, in regard to the buyer, if the average gain i.e $(1/n)\sum_{i=1}{}^n E(R_i)$ is high and the square of the standard deviation quite low i.e k_t $\epsilon(X,\ d)$ *belongs to* R^{n+} yields, $p(k_t)\epsilon(X',d')$ *belongs to* R^{n+}, from the household point of view, with the financial asset quantity $k_t=(k_t^i)_{1\leq i\leq n}$ *belongs to* X there exist random gains associated, $(R_{ti}^k)_{1\leq i\leq n}$ with probabilities $p^k=(p_i^k)_{1\leq i\leq n}$ that an average positive gain, $E(G^k)=\sum_{i=1}{}^n(p_iR_{ti}^k)$ that the household needs to be as high as possible exist, in contrast, $Var(G^k)=(1/n)\sum_{i=1}{}^n(R_{ti}^k\text{–}E(G^k))^2$ is desired to be close as possible to 0. For $n=2$, $k_t=(k_t^1,k_t^2)$ generating the respective random gains, $w_{t1}{}^k=E(R_{t1}{}^k)$ and $w_{t2}{}^k=E(R_{t2}{}^k)$ with probabilities, $p=q=1/2$ where $w_{t1}{}^k=0F$ if $p=1/2$ and $w_{t2}{}^k=100F$ if $q=1/2$, indeed,

$E(G^k)=0(1/2)+100(1/2)=50F$, $\quad Var(G^k)=[(0)^2/2+(100)^2/2]\text{-}(50)^2=10000\text{-}2500=75000$ indeed, $\sigma(G^k)=86,60$

3. The Asset Prices Growth Rate

Lemma 4: *financial asset prices growth rate,* $g^{p(k)}=gop(k_t)$ *exist and follow a normal probability distribution such that,* $g^{p(k)}\approx N(m_{g(p(k))},\ \sigma^2{}_{g(p(k))})$*, thus is also stable*

See the appendix for proof

Proposition 2: *the asset prices growth rate,* $g^{p(k)}$ *yields multiple estimable and testable equilibria*

Proof: by lemma4, $g^{p(k)}$ follow $N(m_{g(p(k))},\ \sigma_{g(p(k))}{}^2)$ i.e $E(g^{p(k)}))=E(\Sigma_{i=1}{}^N\{\Sigma_{j=1}{}^J Y_t^{ij}\})=m_{g(p(k))}>0$ and $Var(g^{p(k)})=1/(\beta\sigma_k)^2=\sigma^2{}_{g(p(k))}$, indeed, asset prices growth rate, $g^{p(k)}$ yields multiple equilibria i.e first, since $\Sigma_{i=1}{}^N\beta_i=1$ where $\beta_i=\beta_j=\beta$ *and* $i\neq j$ *then* $\Sigma_{i=1}{}^N\beta_i=N\beta,\ (i,j)\epsilon\{1,2,\ldots,N\}$ thus $E(g^{p(k)})\rightarrow 0$ and $Var(g^{p(k)})\rightarrow 0$ when $N\rightarrow\infty$ then asset prices growth rate is stationary i.e move at a constant rate, thus, financial assets of the portfolio are not risky. Indeed, if $p(k_{t+1})=p(k_t)\neq 0$ thus asset prices volatility growth rate,$Var(g^{p(k)})\rightarrow 0$, yields $g^{p(k)}$ is bounded. Consequently, assets are attractive for the buyers who are the households. Moreover, since speculations don't make prices increase caused by low asset prices volatility, then the financial asset market is deeply i.e is endowed of many operators since risk is almost absent. Second, if $N\rightarrow 0$ then $E(g^{p(k)})\rightarrow\infty$ thus, gain evolution in growth rate of financial assets is great for the issuers i.e the firms added to the volatility growth rate, $Var(g^{p(k)})\rightarrow\infty$. Moreover, since $g^{p(k)}\ \rightarrow 1$ which is a maximum, then $p(k_{t+1})>p(k_t)$ because of anticipations making the bubbles increase a lot, thus, yield high economic disturbances, indeed, asset price growth rate, $g^{p(k)}$ grow without bound as the bubbles explode, thus yield the economic system to the undeterminacy since equilibrium as well as its stability can't be established. Finally, if $0<N<1$ i.e in the short run, $E(g^{p(k)})\rightarrow l\epsilon]0,1[$, therefore, the asset prices growth rate function, $g^{p(k)}$ converge to a finite positive limit, $l'>0$ such that, $l'<l$ yields $d(p(k_{t+1}),p(k_t))<l$ meaning that, $gain\rightarrow E(gain)$ and $gain(volatility)=Var(gain)<l$, the equilibrium is reached, since, speculations on future asset prices decrease, thus risk neutral households keep entering in the financial market to invest in financial assets.

Proposition3: *both theoretical and empirical links exist between financial assets demand,* $k_t=(k_t^i)_{1\leq i\leq n}$, *their prices,* $p(k_t)$ *and the assets' prices growth rate,* $g^{p(k)}$

Proof: empirically, the previous analysis has shown that, each financial asset $i\epsilon\{1,2,\ldots,n\}$ is endowed of a price, $y_t^i=p(k_t^i)_{1\leq i\leq n}$ that yield expectation and volatility respectively, $E(p(k_t^i))$ and $Var(p(k_t^i))$ associated to risk where the prices volatility growth rate, $Var(g^{p(k)})$ whatever be its nature i.e the link among them may be bijective, injective or surjective. Since, $p(k)$ is defined on a metric space, $(X_{p(k)},d_{p(k)})$ and take it values on the other metric space, $(X_{g(p(k))},d_{g(p(k))})$ *where* $X_{p(k)}$, $X_{g(p(k))}$ *belongs to* R^{n+}, then a given $p(k_t^i),p(k_t^j)\epsilon X_{g(p(k))}$ $i\neq j$ is such that, $d_{p(k)}(p(k_t^i),p(k_t^j))<\delta_\varepsilon$ yields $d_{g(p(k))}(g^{p(ki)},g^{p(kj)})<\varepsilon_\delta$ where $g^{p(ki)}$, $g^{p(kj)}\epsilon X_{g(p(k))}$ is in algebra defined such that, $g^{p(k)}=(p(k_{t+1})-p(k_t))/p(k_t)\epsilon X_{g(p(k))}$ i.e each $i\epsilon\{1,2,\ldots,n\}$ yields $g^{p(ki)}=(p(k_{t+1}^i)-p(k_t^i))/p(k_t^i)=dp(k_t^i)/p(k_t^i)$ $i\epsilon\{1,2,\ldots,n\}$, then the aggregate associated volatility is given by, $Var(g^{p(k)})=Var(dp(k_t)/p(k_t))=Var\{\int_a^b g^{p(k)}dt\}=clog\{Var(p(k_t))\}=clog\{\sigma^2_{p(k)}\}Var(p(k_t))\}=Y_t$ since g^p is an escalator function i.e continuous in each constant ranges $\{[k_t^i,k_{t+1}^i[\}i\epsilon\{1,2,\ldots,n\}$, then, $g^{p(ki)}=(\sigma_i)_{1\leq i\leq n}$ where $U\{[k_t^i,k_{t+1}^i[\}_{1\leq i\leq n}=[a,b]$ thus, the equivalency between $\sum$ and $\int$ yields, $\int_a^b\{gop(s)\}ds\approx\sum_{i=1}^n(\sigma_i)\leq M_\sigma$. Therefore, $\int_a^b Var\{g^{p(s)}\}ds\approx\sum_{i=1}^n Var(\sigma_i)\leq Var(nM_\sigma)=n^2Var(M_\sigma)=0$ because, M_σ is a constant, indeed, $\int_a^b Var\{g^{p(s)}\}ds\rightarrow 0$ i.e $Var\{g^{p(s)}\}\rightarrow 0$ thus, $g^{p(k)}\rightarrow E(g^{p(k)})$ i.e $g^{p(k)}$ admits a fixed point, $g^{p(k^*)}=p(k^*)$ thus, $p(k^*)=k^*$ if p is bijective, then $g^{p(k^*)}=p(k^*)=k^*$ is in that case such that, $g^{p(k^*)}=k^*$ meaning that, a direct link can be established between financial assets demand and prices as well as their growth rate through the functions' relationships. Otherwise, if p is not bijective i.e $p(k^*)\neq k^*$ it yields, k^* may not exist and its stability not ensured since $p^{-1}(k^*)\neq k^*$ i.e the testable asset prices equilibrium, $p(k^*)$ may not exist, thus unable to be reached since, convergence toward a finite limit or a known law is not establish, indeed there may not exist a link with $g^{p(k^*)}$ and $p(k^*)$. Otherwise, if $(p(k_t^i))_{1\leq i\leq n}$ converge toward a limit, $p(k)$ then $g^{p(k)}$ is continuous and converge to a finite limit too because of the existing link among them, thus, several cases yield for a fixed, k_t set: *first,* if $g^{p(k)}$ is an increasing function inside the definition domain, then $(p(k_t^i))_{1\leq i\leq n}$ is also an increasing function. Otherwise, if $g^{p(k)}$ is a decreasing function inside the definition set, then $g^{p(k)}$ is a decreasing function where the under sequences extract from $(p(k_t^i))_{1\leq i\leq n}$ i.e $(p(k_t^{2i}))_{1\leq i\leq n}$ and $(p(k_t^{2i+1}))_{1\leq i\leq n}$ are respectively increasing and decreasing, thus converge to different limits, l and l' then, if $l\neq l'$, the sequences, $(p(k_t^i))_{1\leq i\leq n}$ don't converge since, $p(k_t)$ and $p(k_{t+1})$ don't converge to the same limit i.e $l\neq l'$ then $g(p(k))\neq p(k)$ otherwise, if $l=l'$, then it is a fixed point, k^* such that, $g^{p(k^*)}=p(k^*)$ or similarly, $k^*=p(k^*)$ because of the double bijective function i.e, $k\rightarrow p(k)$ *and* $p(k)\rightarrow g^{p(k)}$, *thus* k^*, the equilibrium exist and its stability is ensured since, $g^{p(k^*)}=p(k^*)=k^*$ *belongs to* $B(0,r_k)$ where $\mathrm{Sup}\{(k_t^i)\}_{1\leq i\leq n}\rightarrow k^*$ and $r_k\leq 1$. Generalizing the analysis in setting, $\{gk_t^j\ _{1\leq j\leq J}\}=\Sigma_{i=1}^N Y_t^i=\Sigma_{i=1}^N\{\Sigma_{j=1}^J Y_t^{ij}\}$ *where* $1\leq i\leq N$ and $1\leq j\leq J$, thus asset prices growth rate can also be written such that, $Y_t=g^{p(kt)}=\Sigma_{i=1}^N\{\Sigma_{j=1}^J Y_t^{ij}\}$

Where: $Y_t^1=g^{p(kt1)}=gop(k_t^1)=(Y_t^{11},Y_t^{12}, Y_t^{1N})$, $Y_t^2=g^{p(kt2)}=gop(k_t^2)=(Y_t^{12},Y_t^{22}, Y_t^{2N})$, $Y_t^J=g^{p(ktN)}=gop(k_t^J)=(Y_{tJ}^{1J},Y_{tJ}^{2J}, Y_{tnI}^{NJ})$. Meaning that, there exist a format (J,N) matrix, since the observation number is, NJ, the i existing average values are given by, $Y_t^i{}^*=\Sigma_{j=1}^J Y_t^{ij}/J$ where the general average theoretical value is, $Y^*=(1/N)\Sigma_{i=1}^N\{(\Sigma_{j=1}^J Y_t^{ij})/J\}$, the general empirical value is then given by, $Y^\wedge=(1/NJ)\Sigma_{i=1}^N\{\Sigma_{j=1}^J Y_t^{ij}{}^\wedge\}$. Since, each asset, i is sold at a quantity, k_t^i then, the prices growth rate, $g(p(k_t^i))=Y_t^i$ and it's estimated value is $Y_t^{\wedge i}=(Y_t^{\wedge i1},Y_t^{\wedge i2},\ldots,Y_t^{\wedge iJ})$ for each asset $i\epsilon\{1,2,\ldots,N\}$, thus, the whole estimated vector is, $Y_t^\wedge=\Sigma_{j=1}^J Y_t^{\wedge i}$. Therefore, it yields, *SCT, SCE* and *SCR* existence in the MCO estimation method that can be written, $SCT=\sum_{i=1}^N\sum_{j=1}^J(Y_{ti}^j-Y^*)^2$, $SCE=\sum_{i=1}^N\sum_{j=1}^J(Y_t^{i*}-Y^*)^2$ with N-1 degrees of freedom and $SCR=\sum_{i=1}^N\sum_{j=1}^J(Y_t^j-Y_t^{i*})^2$ with J-1 degrees of freedom. Since, the tests linked to MCO is the Fisher test, such that, $F=SCE/SCR\approx F(N-1,J-1)$ i.e Fisher Snedecor with $(I-1,J-1)$ degrees of freedom, where, $SCT=SCE+SCR$, then the correlation coefficient, R is expressed such that, $R=(SCE/SCR)^{1/2}$

Since $Y_t^i=(1/J)\sum_{j=1}^J Y_t^j$ such that, $E(Y_t^i)=Y_i^*$ and $Var(Y_t^i)=\sigma^2_{Yi}$, then, $(Y_t^i-Y_i^*)/(\sigma_{Yi}\rightarrow N(0,1)$, therefore, $Y_{ti}\epsilon[-t_{1-\alpha/2}\sigma_{Yi}+Y_i^*;\ t_{1-\alpha/2}\sigma_{Yi}/N+Y_i^*]$. Then, the test consists on, $H_0=\{(1/N)\sum_{i=1}^n(Y_t^i-Y_i^*)^2\leq V^2\}$ for $1-\alpha=95\%$, thus $t_{1-\alpha/2}=F(95\%)$ against, $H_1=\{(1/N)\sum_{i=1}^n(Y_t^i-Y_i^*)^2>V^2\}$. If the statistic test, $F\approx H_0$ then H_0 is accepted at the risk, 5% and if $F_{obs}=1{,}66<F_{4,8}=3{,}84$ then H_0 is rejected at the threshold, $\alpha=5\%$. Otherwise, if we can test the model according to Jarque and Bera test, expressed such that, $\mu_l=(1/N)\sum_{i=1}^n(gop(k_t^i)-gop(k^i{}^*))$, it's statistics given by s is expressed such that, $s=(n/6)S_l^2+(n/24)(L_\mu-3)^2\rightarrow\chi_2^2$ i.e chi-two law with 2 degrees of freedom. Finally, the decision of the adequacy of the model is given by the comparison between the theoretical and the empirical values such that, if they corresponds given the IC (confidence range), estimations and tests yield a prevention tool of the fundamental variables' behavior of the economy over time. More precisely, $E(gop(k_t))=m_{g(p)}$ and

$Var(gop(k_t))=\sigma^2_{g(p)}$ yield, $(gop(k_t)-m_{g(p)})/\sigma_{g(p)} \rightarrow N(0,1)$, then, $gop(k_t)$ *belongs to* $[-t_a\sigma_{g(p)}+\mu_{g(p)};\ t_a\sigma_{g(p)}+m_{g(p)}]=IC(1-t_a)$

Discussions on Volatility and other possible issues

First, if $g^{p(k)}\approx N(m_{g(p)},\ m_{g(p)}\ )$ then if $m_{g(p)}<\infty$ i.e is finite and fixed, but $m_{g(p)}\rightarrow\infty$ i.e is high such that it goes toward infinity, then $IC(1-t_a)\rightarrow\infty$ indeed, $gop(k_t)$ grows without bound and increases prices volatility, since compacity is ruled out and the ball becomes widely open, thus the optimum may no more exist and unable to be reached. In contrast, *if* $0<\sigma_{g(p)}<1$ i.e is low, then, $gop(k_t)$ remains includes inside a compact set i.e a closed ball, thus bounded and admits a finite limit, then the interior solution exists and belongs to the compact set, thus, the optimum is reached. Indeed, households are attracted by the concerned financial assets since their volatility is limited, and the attraction force, depends on the mean of the asset value which is expected to be as high as possible in order to yield higher dividends. The stability of the system is given by the eigenvalues, solutions of the characteristic polynomial solution through their signs such that, when the product is negative and the sum is positive, stability is established. Indeed, since the computation of the characteristic polynomial show that, the matrix viewed before admits two eigenvalues with opposite signs such as the product is thus negative but the sum is positive. Then, we assume that at N, the system is stable i.e the N eigenvalues found are such that, $N/2$ are positive and $N/2$ others are negative such that, the product is negative and in contrast, the sum is positive. Indeed, at the step, $N+1$, it yields, $(N+1)/2$ positive eigenvalues and $(N+1)/2$ others negative eigenvalues with negative product and positive sum, thus stability of the system is ensured. *Second*, if $g^{p(k)}$ and $p(k)$ are convex functions, then $p(k^*)=arg(Min\{g^{p(ki)}{}_{1\leq i\leq N}\})$ whereas, $k^*=arg(\{Minp(k_t^i)_{1\leq i\leq N}\})$ exist, such that $k^*=p(k^*)=g^{p(k^*)}$ where $k^*=Max\{(k_t^i)_{1\leq i\leq n}\}\rightarrow p(k^*)=Min\{p(k_t^i)_{1\leq i\leq n}\}\rightarrow g^{p(k^*)}=Min\{g(p(k_t^i))_{1\leq i\leq n}\}$

4. Finance and Economic Growth

The firms are in competition in the market, their number is n indexed by $j\epsilon\{1,\dots,n\}$. After production done by a given firm j, if an investment opportunity arrives with poison probability $\rho>0$, then firm j invests I_t^j and sells its newly produced capital, $q_tI_t^j$ at the price $q_t>1$ in the capital good market at the end of the period, to buy (sells) additional capital, $k_{1t+1}{}^j-(1-\delta)k_t^j>0(<0)$ before paying out dividends, $d_{t+1}{}^j\geq 0$. Thus, capital sales, q_tI_t and transactions, $q_t(k_{t+1}{}^j-(1-\delta)k_t^j)$ are realized after investment spending, I_t^j thus the firms meet a liquidity mismatch and must access external funds in addition to its internal fund. Indeed, firm j contracts a loan, L_t^j to the bank in order to invest more since internal funds or profits, $\pi_t^j=R_tk_t^j$ are not sufficient for investment due to the liquidity mismatch. The interest rate on the intra-temporal debt is 0 and its price is 1 [10]. Therefore, for a fraction $(1-\rho)$ of the n firms which don't meet an investment opportunity, firms buy (sell) additional capital, $(k_{t+1}{}^j-(1-\delta)k_t^j)>0(<0)$ in the good market at the price, $q_t>1$ and pays dividends, $d_t^j\geq 0$ at the end of the period.

Lemma 5: *since the capital price,* $q_t>1$*, then the optimal investment level near the steady state, is expressed by equation* (13), i.e

$$I_t^j=\pi_{t+1}{}^j+\varepsilon(1-\delta)q_tk_t{}^j+b_{t+1}/1+r_t \tag{13}$$

(see [10] for proof)

Lemma 6: *the stochastic stable long-run estimable and testable economic growth rate,* g^k *follow a normal distribution law i.e* $g^k\approx N(m_{g(k)},\ \sigma^2_{g(k)})$ *where,* $m_{g(k)}=(q_{t+1}-q_t)m_k$ *(see the appendix for proof)*

Proposition 4*: the stochastic stable long-run estimable and testable economic growth rate,* g^k *yields multiple equilibria*

Proof: since, $g^k\approx N(m_{g(k)},\ \sigma^2_{g(k)})$ where $m_{g(k)}=(q_{t+1}-q_t)m_k$, if per-capita physical capital cost is such that, $q_{t+1}=q_t$ then, $E(g^k)=m_{g(k)}=(q_{t+1}-q_t)m_k=0$ thus the economic growth cease since $g^k\rightarrow 0$ because of decreasing marginal productivity of capital, making poor countries grow faster than rich countries ([2]; [3]), the country is lower-middle income country, thus in transition toward market based economy. *Second* if per-capita physical capital cost, q_t is a decreasing function such that $q_{t+1}<q_t$, for all $t\geq 0$ then, $E(g^k)\rightarrow-\infty$ mean that, growth keeps decreasing and may yield to a serious crisis and the concerned country is under developed. *Otherwise*, if per-capita physical capital cost is an increasing function such that, $q_{t+1}>q_t$ for all $t\geq 0$ then $E(g^k)\rightarrow+\infty$, thus the economic growth rate is unbounded since it faces increasing returns and long-run growth ([6]), the country is industrialized. Finally, if $E(g^k)\rightarrow M<+\infty$ where $M>0$, then it is an emerging country. Consequently, in all those cases, volatility level caused by asset prices jump, is an increasing function of the country's development level. Since $Var(g^k)=\sigma^2_{g(k)}$ is associated with the economic growth rate, $g^k=k_{t+1}/k_t-1$ then, if per-capita capital is constant i.e $k_t=k_{t+1}$ it yields $Var(g^k)\rightarrow 0$ and $E(g^k)\rightarrow 0$ in poor countries where finance is not developed yet, thus investments are too low to make growth record gains. Finally, if per-capita capital is an increasing function, i.e $k_{t+1}\geq k_t$ then because of increasing returns, $E(g^k)\rightarrow\infty$ and if $Var(g^k)<M$ where $M<\infty$ because of financial development, then multiple equilibria exist in real economy but, *stability* of the economic system remains. In conclusion, the economic growth rate, g^k is stable, in contrast to financial asset prices growth rate, $g^{p(k)}$ is not stable.

Theorem 1: *the equilibrium in development level, is a mixture* *of finance and economy* $D^*=(k^*,p(k^*),g^{p(k^*)},g^k{}^*)$ *where* $D^*\epsilon]Min\{D\},Max\{D\}[$ *is the locus on the space where, financial development and real economy interaction, yields increasing returns explaining partly development heterogeneities among countries across the world caused by investments differentials.*

Proof: development, D is a function defined from the set E to the set G where E,G *belong to* R^n such that, D admits a threshold, $D^*=(k^*, p(k^*),g^{p(k)}{}^*,g^k{}^*)$ where $D\epsilon]Min\{D\}, Max\{D\}[$ is an escalator function such that, at each range, $\{]a_m,b_m[\}_{m\epsilon N}$ and also yields a development level, $D_m=(k_m,p(k_m),g^{p(km)},g^{km})$, since D is an increasing constant function at each range. Indeed, for each mϵN i.e a given country, if we have $D(g_m)<D^*$ then, the country is under the equilibrium in development level threshold i.e the country is underdeveloped because it as not reached development frontier yet where development is a classified variable such that, $D_1<D_2<\ldots<D_m$ *thus* $D(g_m)>D^*$ and $D(g_m)\rightarrow Max\{D\}$ at the same time, mean that, the country is industrialized and as crossed the development level threshold due to continuous investments caused in the financial market such as credit, financial assets, loans, transactions, money,…. Whereas in contrast, if $D(g_m)=D^*$ then, the country had reached the development level, thus may keep growing or regress assimilated to middle income in the second case and to emerging countries in the first case. Finally, if $D(g_m)<D^*$ such that, $D(g_m)\rightarrow Min\{D\}$ then, the country as not reached development frontier and still far from the threshold, like poorest countries since both financial development and unemployment as well as inflation and debt still high.

In the following part, we extend the previous analysis, in including human capital as an additional investment component inside the growth model with financial assets and bubbles. Doing so, makes human capital becoming an asset like stocks for example in the economy, then human capital is also endowed of an uncertainty character or prices volatility. Indeed, there exist the degree of substitution between human capital and the other financial assets in the consumer's portfolio where the same thing holds for the firms' portfolio in the concern of financial assets issued and dividends given to the financial assets' holders as well as production of goods and services sold, a part of real economy (see figure 2 below).

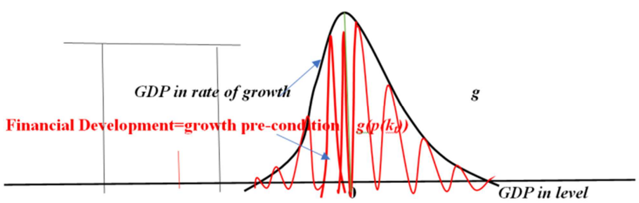

Figure 2. financial development and growth.

5. Global Growth Theory with Human Capital

5.1. Motivations of the Analysis

The previous study introduced financial economics through assets and bubbles in the endogenous growth theory with physical capital accumulation and we find that, *first,* both asset prices growth rate, $g^{p(k)}$ and the economic growth rate, g^k yield multiple estimable and testable equilibria where in the long-run, only real growth, g^k remains stable in contrast, financial assets' growth, $g^{p(k)}$ is not because of anticipations on future asset prices, causing volatility, thus admits multiple unstable and testable equilibria. In the model, since we standardized the skill labor wage rate income to, w^*, then, human capital is also an asset endowed of an uncertainty price. Thus k_t (per-capita physical capital) obtained through consumption can also be converted in capital goods, financial assets as well as human capital, thus are perfect substitute of the household both in the utility function and guide his investment strategy. Indeed, human capital price i.e the direct and the indirect cost of education fluctuates and yields volatility because of technological change speed. Moreover, human capital investment is then, randomly determined since it follow a uniform probability law inside $[0,a^*]$ where $a^*<1$, thus is uncertain and partly join the microeconomics models of education, specifically when incentives to invest in human capital accumulation yields a comparison between cost and benefit.

5.2. Presentation of the Basic Model

In an OLG model, we postulate that, there exist L young workers in total at each period, each worker is endowed of a skill level, h ranging from 0 to infinity. We suppose that, total time possessed by an agent endowed with human capital level, h is normalized to 1 where workers devote a fraction $u(h)$ of his time to current production, whereas, the trained workers also devotes, $1-u(h)$ of his time to human capital accumulation, in the both cases, the rest of time is devoted to leisure. Indeed, production of output is a function of both physical and human capital expressed by equation (14) i.e

$$Y_t=F(K_t,L_t,h_t)=AK_t^{\alpha}(u_th_tL_t)^{1-\alpha} \tag{14}$$

Where, $A>0$ is a non rival knowledge freely accessible (Romer, 1990) assumed to be not remunerated, the parameter of elasticity of physical capital, α is such that, $0<\alpha<1$. Profit maximization yields the respective wage rate income and the interest rate such that

$$w_t=(1-\alpha)Ak_t^{\alpha} \tag{15}$$

$$1+r_t=R_t=\alpha Ak_t^{\alpha-1} \tag{16}$$

Where, $k_t=K_t/u_th_tL_t$ is per-capita intensive capital and according to the Miao-Wang (2018) theory, per-capita physical capital accumulation is expressed by equation (17) ie

$$k_{t+1}=(1-\delta)k_t+\rho(q_tk_t+b_t) \quad (17)$$

Lemma 7: since the capital cost, q_t is such that, $q_t>1$ and human capital return rate, w* is such that, w*>0, then the equilibrium variables, (B_t,q_t,k_t, h_t) satisfy the differential equations, (18) and (19) i.e

$$k_{t+1}-k_t=-\delta k_t+\rho(q_tk_t+b_t^k) \quad (18)$$

$$h_{t+1}-h_t=\zeta h_t(1-u_t)+\phi(w_*h_t+b_t^h) \quad (19)$$

Where $k_0>0$ and $h_0>0$ are given, ϕ follow a uniform probability law inside *[0,a*] where, a*<1,* B_t^k and B_t^h are the respective bubbles' components of physical and human capital accumulations

Proof:first, by the investment definition and lemma5, we have, $I_t=k_{t+1}-(1-\delta)k_t^j=\pi_{t+1}+q_t(1-\delta)\varepsilon k_t+b_{t+1}/1+r_t$. Since, $\pi_{t+1}=0$ at the equilibrium, it yields the differential equation of per-capita physical capital accumulation, equation (20) i.e

$$k_{t+1}-k_t=-\delta k_t+\rho(q_tk_t+b_t) \quad (20)$$

Where k_0 is the given initial per-capita capital, $p(k_t)=q_tk_t+b_t$ is asset prices equation endowed of a fundamental component plus the bubble. By ([10]) written such that, $I_t=h_{t+1}-(1-\zeta(1-u_t))h_t$ indeed, $I_t=\pi^*+\phi(w^*h_t+b_{t+1}{}^h/1+r_t)$. Since the young personal funds are, $\pi^*=0$ at the equilibrium, it yields the differential equation of human capital accumulation expressed such that, $h_{t+1}-h_t=\zeta h_t(1-u_t)+\phi(w_*h_t+b_t^h)$ where $h_0>0$ is given and $w^*>0$ is the benefit of education or the return rate of education, b_t^h is human capital accumulation bubbles. Indeed, human capital accumulation can be written such that, (21) i.e

$$h_{t+1}=\zeta(1-u_t)h_t+\phi(w_*h_t+b_t^h) \quad (21)$$

Where, $\zeta>0$ is human capital productivity parameter, ϕ is the fraction of educated young agents which follow a uniform probability law inside *[0,a*]* where *a*<1*

Corollary1*: according to lemma4, the respective per-capita physical and human capital accumulations epressed such that,* $k_{t+1}=\beta_kk_t+\mathcal{E}_{tk}$ *and* $h_{t+1}=\beta_hh_t+\mathcal{E}_{th}$ *are testable,*

Proof: $(1-\delta)k_t+\rho q_tk_t=(1-\delta+\rho q_t)k_t=\beta_kk_t$, $\mathcal{E}_{tk}=\rho B_t^k$, $\beta_hh_t=(1+\zeta(1-u_t))+\phi w_*)h_t$ and $\mathcal{E}_{th}=\phi B_t^h$ such that, $\mathcal{E}_{tk}\approx N(0, \sigma_k^2)$ and $\mathcal{E}_{th}\approx N(0,\sigma_h^2)$, $\sigma_k^2=(\rho\sigma)^2$ and $\sigma_h^2=(\phi\sigma)^2$, $\zeta>0$ is human capital productivity parameter, ϕ is the fraction of educated young agents which follow a uniform probability law inside *[0,a*], a*<1*. Setting, $p(k_{t-1})=k_t$ yields, $k_t=\theta_kk_{t-1}+\mathcal{E}_{t-1k}$ where $k_t=(k_t^i)_{1\leq i\leq n}$ indeed, $E(k_t)=k^*=(1/n)\sum_{i=1}{}^nk_t^i$ thus, $k_t\approx AR(1)$ (Arma(1) process) i.e $k_t=(1-\theta_k)k^*+\theta_kk_{t-1}+\mathcal{E}_{t-1k}$ which is equivalent to $k_t=C+\theta_kk_{t-1}+\mathcal{E}_{t-1k}$ where, $C=(1-\theta_k)k^*$. Indeed, $\alpha(L)k_t=\alpha_0k_t+\alpha_1k_{t-1}+....+\alpha_pk_{t-n}$, therefore, $Cov(k_t,k_{t-j})=E(k_t-E(k_t))(k_{t-j}-E(k_{t-j}))=\gamma_k$ for $i\neq j$ and $i,j\epsilon\{1,2,...,n\}$ where the correlation coefficient is, $\rho_k=\gamma_k/Var(k_t)$ Indeed, if ρ_k tends to 0, then variables are not correlated, in contrast, when $\rho_k>0$, then financial assets are correlated. Indeed, if $E(k_t)=E(k_{t-j})=m_k$ then $Cov(k_t,k_{t-j})=Var(k_t)=\sigma^2$ since $k_t^i\approx N(m_k^i,\sigma_i^2)$, then, there exist, t_k such that, $t_k=(k_t^i-m_k^i)/\sigma_i\approx N(0,1)$, therefore, k_t^i *belongs to* $[-t_{1-b}\sigma_i+m_k^i,t_{1-b}\sigma_i+m_k^i]$. Since, $k_t=\cap_{i=1}{}^nk_t^i$ then, k_t *belongs to* $\cap_{i=1}{}^n[-t_{1-b}\sigma_i+m_k^i,t_{1-b}\sigma_i+m_k^i]$ is thus, closed and bounded, indeed converge to a finite limit. Otherwise, if $k_t=\dot{U}_{i=1}{}^nk_t^i$ then, k_t *belongs to* $\dot{U}_{i=1}{}^n]-t_{1-b}\sigma_i+m_k^i, t_{1-b}\sigma_i+m_k^i[$ is thus, open and unbounded, indeed diverge to ∞.

Proposition5: *the respective per-capita physical and human capital asset prices functions,* $p(k_t)$ *and* $p(h_t)$ *defined on the metric spaces,* $((E_{1i})_{i=k,h},(d_i)_{i=k,h})$ *and take values on* $((G_{1i})_{i=k,h},(d_i')_{i=k,h})$ *such that,* $p(k_t)=q_tk_t+\mathcal{E}_t^k$ *and* $p(h_t)=w_th_t+\mathcal{E}_t^h$ *are empirically testable*

Proof: since physical capital stock prices definition is given by, $p(k_t)=q_tk_t+\mathcal{E}_t^{k^*}$, thus, is a function which takes it values from $k_t\,\epsilon E_{1k}$ to $p(k_t)\,\epsilon G_{1k}$. Since the function is include inside a metric space, it admits a convergence sequences to a unique limit, thus is from Cauchy, then ensures the continuity of $p(k_t)$ inside the whole set, E_{1k} i.e the function $p(k_t)=q_tk_t+\mathcal{E}_t^k$ where $\mathcal{E}_t^k\approx N(0,\sigma_k^2)$. In parallel, since human capital stock price, $p(h_t)$ is a continuous function from E_{1h} to G_{1h}, then by definition, its expression is given by $p(h_t)=w_th_t+\mathcal{E}_t^h$ where $\mathcal{E}_t^h$ follow $N(0,\sigma_h^2)$. The condition for the asset prices functions gap to converge in order to ensure the economic stability, is the existence of a threshold in bubbles term, $\mathcal{E}^*$ which render, $\{p(k_t)-p(h_t)\}$ converge to 0 i.e a locus on the space, where, the both asset prices functions meet on the space. That locus is provided when the speed of the dynamics of the prices functions difference, equals 0 i.e when $\partial(p(k_{t+1})-p(h_{t+1}))/\partial t=0$, ensuring the stability of the equilibrium, whereas, its existence is given by the following equality i.e $p(k_t)=p(h_t)$ thus the average white noise, $\mathcal{E}^*=(\mathcal{E}_t^h+\mathcal{E}_t^k)/2$, yields the sequences, $\{p(k_t)-p(h_t)\}_{t\geq 0}$ converge to a finite limit, l since $p(k_t)$ speed i.e $\partial p(k_t)/\partial t$ is reduced to the minimum and when, $\partial p(k_t)/\partial t=0$ then, p^* as a fixed point i.*e* $p(k^*)=k^*\leq 1$ where $p(k^*)>0$. Since $z>0$ two effects are in play, *first*, capital cost, q_t increases the bubbles, whereas, human capital cost, w^* make the bubbles decrease, indeed, w^* is not a financial variable able to generate a shock on real economy through financial economics. Consequently, in the long run, an equilibrium may emerge, since human capital and physical capital costs, w^* and q_t are stable.

Corollary 2: *the respective estimable and testable physical and human capital asset prices growth rates,* $g^{p(k)}$ *and* $g^{p(h)}$ *exist (see the appendix for proof)*

Corollary3: *the aggregate asset prices growth rate,*

$g^{p}=g^{p(k)}+g^{p(h)}$ *is empirically space viewable*

Proof: since $g^{p(k)}$ and $g^{p(h)}$ are independent, identically distributed and follow *a normal distribution,* thus, $Cov(g^{p(k)}, g^{p(h)})=0$ yields, $E(g^{p})$ and $Var(g^{p})$ exist and are expressed such that, $E(g^{p})=E(g^{p(k)})+E(g^{p(h)})=q_{t-1}m_k + w_{t-1}m_h = m_{g(p)}$ and we have in parallel, $Var(g^{p})=Var(g^{p(k)})+Var(g^{p(h)})=\sigma^{g(k)2}+\sigma^{g(h)2}=(\sigma^{g(p)})^2$. Consequently, $g^{p}\approx N(m_{g(p)},\sigma^{g(p)2})$ Indeed, the space location of g^{p} consists on viewing: *first,* that, $(g^{p}-m_{g(p)})/\sigma^{g(p)} \rightarrow N(0,1)$ thus, at the threshold, a', it yields g^{p} *belongs to* $[-q_a\cdot\sigma^{g(p)})+m_{g(p)},m_{g(p)}+q_a\cdot\sigma^{g(p)}]$ i.e $P(g^{p}$ *belongs to* $[-q_a\cdot\sigma^{g(p)})+m_{g(p)},m_{g(p)}+q_a\cdot\sigma^{g(p)}])=1-a'$ *indeed,* $P(A<g^{p}<B)=\int_A^B(2\pi)^{1/2}exp\{-(x)^2/2\}dt=1-a'$ where, $A=-q_a\cdot\sigma^{g(p)})+m_{g(p)}$ and, $B=q_a\cdot\sigma^{g(p)})+m_{g(p)}$

Assumption 4: $x\approx N(m_x,\sigma_x^2)$ *and* $z\approx N(m_z,\sigma_z^2)$

Proposition6*: the respective economic growth rates of per-capita physical capital and human capital* g^{k} *and* g^{h} *expressed such that,* $g^{k}=\beta^{g(k)}x+\mathcal{E}^{g(k)}$ *and* $g^{h}=\beta^{g(h)}z+\mathcal{E}^{g(h)}$ *are empirically estimable. Where,* $\mathcal{E}^{g(k)}=b_t^{k}*=(\rho/(k_0)^t)b_t^{k}$ *defined from* E^{k} *to* F^{k} *with* E^{k} *and* F^{k} *belongs to* R^{n}, $\mathcal{E}^{g(h)}=b_t^{h}*=(\phi/(h_0)^t)b_t^{h}$ *defined from* E^{h} *to* F^{h} *with* E^{h}, F *belongs to* R^{n}

Proof: let g^{k} and g^{h} *be* defined on a metric space, $((E^{i})_{i=k,h},(d^{i})_{i=k,h})$ and takes it values inside $(F^{i})_{i=k,h},(d^{i\prime})_{i=k,h}$) which is also a metric space, such that the growth rates expressions are respectively like follow $g^{k}=(-\delta+\rho q_t)+b_t^{k}=\beta^{g(k)}x+\mathcal{E}^{g(k)}$ and $g^{h}=(\zeta(1-u_t)+\phi w_{*}-1)+b_t^{h*}=\beta^{g(h)}z+\mathcal{E}^{g(h)}$ In order to prove proposition6, let us see, *first,* $g^{k}=\sum_{i=0}^{n}[(-\delta_i+\rho_i q_{ti})+e_i b_{ti}]=\sum_{i=0}^{n}[\rho_i q_{ti}+(e_i b_{ti}-\delta_i)]=\beta^{g(k)}x+\mathcal{E}^{g(k)}$ indeed, $g^{k}=\beta^{g(k)}x+\mathcal{E}^{g(k)}$ where $\beta^{g(k)}=(\rho_i)_{1\leq i\leq n}$ *and* $x=(q_{ti})_{1\leq i\leq n}$. therefore, $E(g^{k})=\beta^{g(k)}m_x$ i.e $E(g^{k})=m_{g(k)}$ and $E(g^{h})=\beta^{g(h)}m_z=m_{g(h)}$ by assumption5. *If* $E(g^{k})=m_{g(k)}<E(g_h)=m_{g(h)}$ then there exist $p>0$ such that, $m_{g(k)}-p=m_{g(h)}$ indeed, there also exist the aggregate growth rate, g such that, $g=g^{k}+g^{h}$ where, $E(g)=E(g^{k}+g^{h})=m_{g(h)}(\beta^{g(k)}+\beta^{g(h)})+p\beta^{g(k)}$ and $\mathcal{E}^{g(k)}=(e_i b_t^{i}-\delta_i)_{1\leq i\leq n}$. Then, $E(\mathcal{E}^{g(k)})=E(\Sigma_{i=1}^{n}(e_i b_t^{i}-\delta_i))=\Sigma_{i=1}^{n}(-\delta_i)=-n(n+1)/2$ *if* $(\delta_i)_{1\leq i\leq n}=i$ therefore, $E(\mathcal{E}^{g(k)})=-n(n-1)/2\rightarrow-\infty$ since by the assumption, $\mathcal{E}^{g(k)})\geq0$ it yields, $E(\mathcal{E}^{g(k)})\rightarrow0$ indeed, $Var(\mathcal{E}^{g(k)})=\sigma^2$ thus, $\mathcal{E}^{g(k)}\approx N(0,\sigma^2)$, there thus exist, $T_{g(k)}=(\mathcal{E}^{g(k)}/\sigma\approx N(0,1)$. Consequently, the usual regression analysis method can be applied. Note that, for the empirical theory to be applied in that context, we need to ensure of several properties, which in the both cases are the same i.e: *first,* the relationship between exogenous and endogenous variables must be linear, according to the observations of the exogenous variables. *Second,* there must not be any correlation among the error term (exogeneity of variables). *Third,* perturbations variables must not be linked to exogenous variables (homoscedasticity). *Fourth*, the error terms' square of the standard deviation must be stable over time i.e σ^2.must remain stable over time *Finally,* the error term, $\mathcal{E}$ must follow a centered normal law with a stable variance, σ^2, we are in presence of homoscedasticity. Otherwise, if σ^2 is variable, we are subject to the heteroscedasticity case. Consequently, the both previous models can be written in terms of matrices, where *first,* $g^{k}=\beta^{g(k)}x+\mathcal{E}^{g(k)}=X_1{}^t\beta_1+X_2{}^t\beta_2+\mathcal{E}^{g(k)}$ can be written such that, $AX=B$ i.e

$$\begin{pmatrix} X1'X1 & X1'X2 \\ X2'X1 & X2'X2 \end{pmatrix}\begin{pmatrix} '\beta 1 \\ '\beta 2 \end{pmatrix} = \begin{pmatrix} X1'Y \\ X2'Y \end{pmatrix}$$

By the Firsch and Waugh method, we can find a correlation between estimators, $\beta^{g(ki)}$ and $\beta^{g(kj)}$ such that, ${}^t\beta^{g(k1)}=(X_1{}^tX_1)^{-1}[{}^tX_1Y]-({}^tX_1X_1)^{-1}[{}^tX_1X_2]\beta^{g(k2)}$ and when compare to the MCO method, where $\beta^{g(k)}=(\beta^{g(k1)}, \beta^{g(k2)})$, we obtain its estimator, $\beta^{g(k)}*$ such that, $\beta^{g(k)}*(X\ {}^tX)^{-1}X^ty$ since, $X=(X_1,X_2\ )$ thus, $E(\beta^{g(k)}*)=\beta^{g(k)}$ and $Var(\beta^{g(k)}*)=(X^tX)\sigma$, i.e $\beta^{g(k)}*\approx N({}^t\beta^{g(k)}, (X^tX)\sigma)$, then allow for tests elaboration. Indeed, given, g^{k} a $(nxn, 1)$ matrix, X a (nxm, K) matrix and $\beta^{g(k)}$ a $(K,1)$ matrix, there exist an invertible variance-covariance matrix, Φ such that, equation (22) i.e

$$\beta^{g(k)}*({}^tX\Phi^{-1}X)^{-1}({}^tX\Phi^{-1}Y) \quad (22)$$

Applying the same process to g^{h} yields the estimator of the model parameters given by equation (23) i.e

$$\beta^{g(h)}*({}^tZ\Phi^{-1}Z)^{-1}({}^tZ\Phi^{-1}W) \quad (23)$$

Because the functions of the growth rates, g^{k} and g^{h} are defined in a metric space, each sequences $k_t=(k_t^{i})_{1\leq i\leq n}$ and $h_t=(h_t^{j})_{1\leq j\leq m}$ converge respectively to finite limits, $l^{g(k)}$ and $l^{g(h)}$

Moreover, since $|-\delta+\rho q_t|\leq1$ and $|\zeta(1-u_t))+\phi w_{*}-1|\leq1$, then, $d_k(k_t^{i},l^{k})\leq\eta_k$ yields, $d_{g(k)}(g(k_t^{i}), g(l^{k}))\leq\varepsilon_k$ thus, $k_t=(k_t^{i})_{1\leq i\leq n}$ converge to l^{k} that yields g^{k} converge to $g(l^{k})$ where l^{k} and $g(l^{k})$ are the respective unique limits of k_t and $g(k_t)$ or the frontier. By the same reasoning, $h_t=(h_t^{j})_{1\leq j\leq m}$ converge to l^{h} that yields g^{h} converge to $g(l^{h})$, a unique limit. Thus, the equilibrium, (g^{k*},g^{h*}) is locally stable.

Proposition 7: *the aggregate stochastic endogenous economic growth rate,* $g=g^{k}+g^{h}$ *exist and is stable (see the appendix for proof)*

Summary of the Extended Economic Growth Theory: "*global long-run growth",* $g*$ expressed by the sum of g^{p} and g *i.e* $g*=g^{p}+g$ exist,

Where $g^{p}=g^{p(k)}+g^{p(h)}=g^{p(h)+p(k)}=g^{p(h+k)}$ is aggregate asset prices growth rate whereas, $g=g^{k}+g^{h}$ is the aggregate economic growth rate. Indeed financial development, g^{p} is what causes impacts on real economic growth rate, g measured by the *GDP* rate of growth. Since the both growth rates i.e g^{p} and g are such that, one is volatile and the other is stable, then the conjunction of the both yields the unique

Pareto optimal empirical equilibrium in growth rates, g^*, "global long-run growth" of a given country is necessary in order to understand economic dynamics movements in function of financial development to explain the countries power in economics i.e R&D and technological change even driven through exchange trade.

Note that, $g^{p(k)}=(g(p(k_{t+1}))-g(p(k_t)))/g(p(k_t))=g(p(k_{t+1}))$ and $g^{p(h)}=(g(p(h_{t+1}))-g(p(h_t)))/g(p(h_t))=g(p(h_{t+1})$.

Thus, the whole yields global economic growth rate, $g^*=g^k+g^h$ i.e the sum of physical capital and human capital growth rates, where $g^k=(I_{t+1}-I_t)/I_t$ following ([10]), growth is respectively defined as the rate of the difference between investment accumulation, $I_{t+1}-I_t$ and its current level, I_t at time t and the same thing for human capital i.e $g^h=(g(h_{t+1})-g(h_t)))/g(h_t)$.

6. Conclusion

We have joined finance and economic growth where the summary of the both variables, yields "global long-run growth" that can be put in control, since in a given country, economic power depends on financial development capacity to satisfy investment opportunities, thus the financial aspect of the economy is a departure of increasing returns. However, finance part of the economy is difficult to anticipate because of asset prices volatility, thus create disorders even it is crucial for advances in highest growth rates because of its impact on the interest rate. Indeed, since the "global long-run growth" yields multiple equilibria, the Pareto optimality of the equilibrium, must be established first, in order for the global dynamic system to be located and forecast and serve as an economic policy tool for the social planner. The idea is to consider that, volatility yields economic instability, creating crashes as well as booms have a limit, due to the fact that, even if volatility widen a lot, global economy (the sum of real and finance growth rates assets prices) remains stable as long as it is includes in the compact set like a ball of center 0 and of radius no more than1. If it is not the case, then it should be looked for the neighborhoods system stability, where the optimal cannot be reached since volatility is too high and difficult to measure empirically, but can also be consider at a level of an existing bound, $M=Sup\{Var(g^{p(ki)}{}_{i\epsilon N})\}<+\infty$ such that, around $B(r_M,M)$ where $0<r_M<1$, the equilibrium in asset prices volatility growth rate exist, thus yields, global long-run growth stability and measurability for forecasting. The weakness of the model, comes from the perfect econometrics hypothesis satisfying methods such as regression analysis must be filled which guaranty no technical problems to consider, then data on the sum of assets prices of finance capital and human capital are stationary, thus avoid more sophisticated methods making data stationary before, then "global long-run growth" exist and is stable.

Appendix

Proof of Lemma4: $g^{p(k)}\approx N(m_{g(p(k))}, \sigma_{g(p(k))}{}^2)$ then $g(p(k_t))=(p(k_t)-p(k_{t-1}))/p(k_{t-1})=[(k_t\beta+\mathcal{E}_t)-(k_{t-1}\beta+\mathcal{E}_{t-1})]/(k_{t-1}\beta+\mathcal{E}_{t-1})=[\beta(k_t-k_{t-1})+(\mathcal{E}_t-\mathcal{E}_{t-1})]/(k_{t-1}\beta+\mathcal{E}_{t-1})$. Indeed, taking the logarithm of the function $gop(k_t)$ yields, $Y_t=Log(gop(b_t))=log\{[\beta(k_t-k_{t-1})+(\mathcal{E}_t-\mathcal{E}_{t-1})]/(k_{t-1}\beta+\mathcal{E}_{t-1})\}=log\{\beta(k_t-k_{t-1})+(\mathcal{E}_t-\mathcal{E}_{t-1})\}-log\{k_{t-1}\beta+\mathcal{E}_{t-1})\}=log\{\beta(k_t-k_{t-1})(1+(\mathcal{E}_t-\mathcal{E}_{t-1})/\beta(k_t-k_{t-1})\}-log\{k_{t-1}\beta(1+\mathcal{E}_{t-1}/k_{t-1}\beta\}\approx log\{\beta(k_t-k_{t-1})\}-log\{k_{t-1}\beta\}$;

Since $(1+(\mathcal{E}_t-\mathcal{E}_{t-1})/\beta(k_t-k_{t-1})\approx 1$ and $(1+\mathcal{E}_{t-1}/ k_{t-1}\beta \approx 1$ around 0 by Taylor approximation. Indeed, $E(log\{\beta(k_t-k_{t-1})\}-log\{k_{t-1}\beta\})=logE\{\{\beta(k_t-k_{t-1})\}-logE\{k_{t-1}\beta\}=log(1/\beta m_k)$. Taking now the exponentiel function which is an increasing function, it yields, $E(gop(k))=exp\{-log(\beta k)\}=1/\beta m_k$, therefore, $E(gop(k_t))=1/\beta m_k$, $Var\{Log(gop(k_t))\}=Log(Var(\beta(k_t-k_{t-1}))-Log(Var(k_{t-1}\beta)\}=Log\{1/(\beta\sigma_k)^2\}$, and taking the exponentiel, which is an increasing function, it yields, $Var(gop(k_t))=1/(\beta\sigma_k)^2$. Consequently, the asset prices growth rate, $g^{p(k)}=gop(k_t)$ follows $N((\beta m_{t-1})^{-1},(\beta\sigma_k)^{-2})$ i.e $g^{p(k)}\approx N(m_{g(p(k))}, \sigma_{g(p(k))}{}^2)$ where $m_{g(p(k))}=(\beta m_{t-1})^{-1}$ and $\sigma_{g(p(k))}{}^2=(\beta\sigma_k)^{-2}$ then, there exists, $Y_{g(p)}$ such that, $Y_{g(p)}=(gop(k_t)-(\beta m_k)^{-1})/(\beta\sigma_k)^{-1}\approx N(0,1)$, i.e, $gop(k_t)$ *belongs to* $[-t_{1-a/2}(\beta\sigma_k)^{-1}+(\beta m_k)^{-1};t_{1-a/2}(\beta\sigma_k)^{-1}+(\beta m_k)^{-1}]$ a compact set, thus $gop(k_t)$ is closed and bounded, indeed, converge to a unique finite limit.

Proof of Lemma 6: $g^k\approx N(m_{g(k)},\sigma_{g(k)}{}^2)$

According to the literature, the economic growth rate, g is a linear function defined on Z which take it values on R such that, for any $I_t^j \epsilon Z$ *belongs to* R, *it yields,* $g(I_t^j)=g^k=(I_{t+1}{}^j/I_t^j)-1$. According to lemma3, the economic growth rate, g^k expression is given by, $g^k=(\pi_{t+2}{}^j+\varepsilon(1-\delta)q_{t+1}k_{t+1}{}^j+b_{t+2}/1+r_{t+1})/(\pi_{t+1}{}^j+\varepsilon(1-\delta)q_tk_t{}^j+b_t/1+r_t)-1$ Since at the steady state equilibrium, $\pi_{t+2}{}^j=\pi_{t+1}{}^j=0$ it yields, $g^k=(\varepsilon(1-\delta)q_{t+1}k_{t+1}{}^j+b_{t+2}/1+r_{t+1})/(\varepsilon(1-\delta)q_tk_t^j+b_t/1+r_t)-1$, Indeed, $log([(\varepsilon(1-\delta)q_{t+1}k_{t+1}{}^j+b_{t+2}/1+r_{t+1})/[(\varepsilon(1-\delta)q_tk_t^j+b_t/1+r_t)]-log(e))=log([(\varepsilon(1-\delta)q_{t+1}k_{t+1}{}^j+b_{t+2}/1+r_{t+1})]-log[(\varepsilon(1-\delta)q_tk_t^j+b_t/1+r_t)/e]$, therefore, $log\{E([(\varepsilon(1-\delta)q_{t+1}k_{t+1}{}^j+b_{t+2}/1+r_{t+1})]-log[(\varepsilon(1-\delta)q_tk_t^j+b_t/1+r_t)/e]\}=log\{([(\varepsilon(1-\delta)q_{t+1}k_{t+1}{}^j]/[(\varepsilon(1-\delta)q_tk_t^j/e]\}=log(q_{t+1}k_{t+1}{}^j)/(q_tk_t^j)-ln(e)$ since $b_t\approx N(0,\sigma^2)$ (lemma2). Taking the exponentiel, an increasing function, it yields, $E(g^k)=(q_{t+1}k_{t+1}{}^j)/(q_tk_t^j)-1$, $Var(log(g^k))=Var(log([(\varepsilon(1-\delta)q_{t+1}k_{t+1}{}^j+b_{t+2}/1+r_{t+1})]-log[(\varepsilon(1-\delta)q_tk_t^j+b_t/1+r_t)/e])=log\{Var(([(\varepsilon(1-\delta)q_{t+1}k_{t+1}{}^j+b_{t+2}/1+r_{t+1})]-Var([(\varepsilon(1-\delta)q_tk_t^j+b_t/1+r_t), e)\}=log(Var(b_{t+2}/1+r_{t+1})])=log(Var(b_{t+2}/1+r_{t+1})])=log(\sigma^2[(1/1+r_{t+1})^2)-log(\sigma^2(1/1+r_t)^2)=0$ Then, taking the exponentiel, an increasing function, it yields, $Var(g^k)=(log(exp\{0\})=0$, thus

$E(g^k)=(q_{t+1}k_{t+1}{}^j)/(q_tk_t^j)-1$ Since, $\log\{(q_{t+1}k_{t+1}{}^j)/(q_tk_t^j)-exp\{0\})\}=\log(\{(q_{t+1}k_{t+1}{}^j))-\log(q_tk_t^j)$ because of linearity, indeed $E(ln(g^k))=E(ln(q_{t+1}k_{t+1}{}^j))-E(\ln(q_tk_t^j))$, taking the exponentel, an increasing function, it yields, $E(exp\{ln(g^k)\})=q_{t+1}m_{t+1}{}^j-q_tm_t^j$ $=(q_{t+1}-q_t)m_k$ since, $m_t^j=m_{t+1}{}^j\rightarrow m_k$ therefore, $E(g^k)=(q_{t+1}-q_t)m_kq_{t+1}m_{t+1}{}^j\ /q_{t+1}m_{t+1}{}^j$ and $Var(g)=0$ $Var(ln(g^k)=Var(log(\{(q_{t+1}k_{t+1}{}^j))-\log(q_tk_t^j))$, taking the exponentiel, an increasing function, it yields, $Var(g^k)=((q_{t+1})^2-(q_t)^2)\sigma^2_k=\sigma^2_g$ consequently, $g^k\approx N(m_{g(k)}=(q_{t+1}-q_t)m_k,\ \sigma^2_{g(k)})$ where $m_{g(k)}=(q_{t+1}-q_t)m_k$

Proof of corrollary2: $g^{p(k)}$ and $g^{p(h)}$ *exist and are both estimable and testable.*

On the one hand, $p(k_t)=q_tk_t+\mathcal{E}_t^k$ where $\mathcal{E}_t^k\approx N(0,\ \sigma_k^2\ )$. Indeed, since by definition, we have: $g^{p(k)}=p(k_t)-p(k_{t-1})/p(k_{t-1})=(q_tk_t+\mathcal{E}_t^k-q_{t-1}k_{t-1}-\mathcal{E}_{t-1}{}^k)/q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k=[(q_tk_t-q_{t-1}k_{t-1})+(\mathcal{E}_t^k-\mathcal{E}_{t-1}{}^k)]/q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k=dq_{t-1}k_{t-1}+d\mathcal{E}_{t-1}{}^k/q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k$ $=d(q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k)/q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k=g^{p(k)}$ therefore, $\int_R g^{p(k)}=log\{q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k\}$ thus, taking the exponentiel, yields, $exp\{\int_R g^{p(k)}\}=q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k$ indeed, $g^{p(k)}=q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k$. On the other hand, we have, $p(h_t)=w_th_t+\mathcal{E}_t^h$ where $\mathcal{E}_t^h$ follow $N(0,\sigma_h^2)$ thus yields, $exp\{\int_R g^{p(h)}\}=w_th_t+\mathcal{E}_t^h$ consequently, $g^{p(h)}=w_{t-1}h_{t-1}+\mathcal{E}_{t-1}{}^{h.,}$ therefore, $E(g^{p(k)})=E(q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k)=q_{t-1}m_k=m_{p(k)}$ thus $Var(g^{p(k)})=Var(q_{t-1}k_{t-1}+\mathcal{E}_{t-1}{}^k)=(q_{t-1})^2(\sigma_k^2)=(\sigma^{g(k)})^2$ thus $E(g^{p(h)})=E(w_th_t+\mathcal{E}_t^h)=w_tm_h$ and $Var(g^{p(h)})=Var(w_{t-1}h_{t-1}+\mathcal{E}_{t-1}{}^h)=(w_{t-1})^2(\sigma_h^2)=(\sigma^{g(h)})^2$. Since $g^{p(k)}\approx N(q_{t-1}m_k,(\sigma^{g(k)})^2)$ and $g^{p(h)}\approx N(w_{t-1}m_h,(\sigma^{g(h)})^2)$, there exist $Z_{g(k)}$ and $Z_{g(h)}$ such that, $Z_{g(k)}=(g^{p(k)}-q_{t-1}m_k)/\sigma^{g(k)})\rightarrow N(0,1)$ *and* $Z_{g(h)}=(g^{p(h)}-w_{t-1}m_h)/\sigma^{g(h)}\rightarrow N(0,1)$. Indeed, both physical and human capital assets growth prices are include in a compact set i.e $g^{p(k)}$ *belongs to* $[-t_{1-a/2}\sigma^{g(k)})+q_{t-1}m_k,t_{1-a/2}\sigma^{g(k)})+q_{t-1}m_k]$ and $g^{p(h)}$ *belongs to* $[w_{t-1}m_h-t_{1-a/2}\sigma^{g(h)},w_{t-1}m_h+t_{1-a/2}\sigma^{g(h)}]$, thus asset prices growth rates, $g^{p(k)}$ and $g^{p(h)}$ are both estimable and testable.

The decision of the household to buy the risky assets or not, consists on the comparison of the values of $E(G^k)$ and of $\sigma(G^k)$ in such a way that, the first value needs to be as high as possible in contrast to the second value. In the previous example, $E(G^k)<\sigma(G^k)$ i.e volatility is lower than the average gain, thus the asset is not risky. In conclusion, at the level of the firms, because of the managers' ability, gain can be unbounded, in contrast to the risk neutral household portfolio choice strategy and can face high future prices anticipations, i.e high volatility, thus quite highly risky specifically if $Cov(k_t^1,k_t^2)=(k_t^1-E(k_t^1))(k_t^2-E(k_t^2))/2$ is highly positive, then the investment is risky for the investor. Otherwise, if $Cov(k_t^1,k_t^2)\rightarrow 0$, then the assets are not risky. Consequently, the household can also choose the least risky asset in his portfolio, since the whole weight$=100\%=1$ i.e the sum of weights of the assets, $\sum_{i=1}^n p_i=1$ if he chooses the least risky asset, the risk neutral households increase. Moreover, for each asset, $i\epsilon\{1,2,\dots,n\}$ of quantity, $(k_t^i)_{1\leq i\leq n}$ *belongs to X* there exist the associated prices $\{p(k_t^i)\}_{1\leq i\leq n}$ *belongs to X'* which yield $E(p(k_t^i))$ and $Var(p(k_t^i)$ as decision variables for the issuers firms and in parallel, households or investors' decision criteria is based on the return rate, $G^k\approx R_t^i$ given asset prices volatility such that, $E(G^k)\rightarrow+\infty$ and volatility, $Var(G^k)=E(G^k-E(G^k))^2\rightarrow 0$ as well as risk neutral households prefer, $Cov(i,j)\rightarrow 0$ for $i\neq j$ for *two* given financial assets. Consequently, *first,* when the asset prices function is inside a compact set i.e,

$$p(k_t) \text{ belongs to } [-t_{1-a/2}\sigma_y+\beta m_k;\ t_{1-a/2}\sigma_y+\beta m_k\ ]=B(0,\varepsilon_k)$$

where $\varepsilon_k=t_{1-a/2}\sigma_y+\beta m_k$ is the radius of the closed ball, thus k_t *belongs to* $[-\sigma_kt_{1-a/2}+m_k;\sigma_kt_{1-a/2}+m_k]$ yields $p(k_t)$ *belongs to* $B(0,\varepsilon_k)$ where $t_{1-a/2}$ is the value given b y the table of $N(0,1)$, then, $p(k_t)$ converge to a finite limit since it is closed and bounded. Thus, households are willing to buy the asset if the average gain yields, βm_k, i.e if $\beta\geq 1$ and $m_k>1$, then $E((p(k_t))=E(Gain)\rightarrow\infty$ i.e converge to infinity, the assets yield a maximum gain to the seller. Therefore, the bubbles collapse, explain, $p(k)$ fall, the economy also falls down. Finally, the bubbles' explosion, yields the economy to an indeterminacy locus $p(k)$ becomes difficult to locate. Indeed, in order to reach the equilibrium, bounds on the bubbles' evolution must be looked for, i.e we must found a given, $M>0$ existence, such that, $M=Inf(p(k_t^i))_{1\leq i\leq n}$. Thus, if $0<M<1$, then for each asset i of quantity, k_t^i, the expectation of its future price yields, $E(p(k_t^i))<M=Min\{p(k_t^i)_{1\leq i\leq n}\}$ for all $i\epsilon\{1,2,\dots,n\}$ thus, $p(k_t)$ converge to a finite limit, since $p(k)$ *belongs to* $B(0,\ M)$. If $M<+\infty$, then $E(p(k_t^i))\rightarrow p(k_t^i)$ i.e for all $\mathcal{E}_M>0$ there exist, $n(\mathcal{E}_M)$ such that, $\{d(k_t^i,k_t^j)_{i\neq j}\}<n(\mathcal{E}_M)$ yields $\{d'(p(k_t^i),p(k_t^j))_{i\neq j}\}<\mathcal{E}_M$. Moreover, if p is an isometric function, it admits a fixed point, $k^*=p(k^*)$, where $k_t=\{k_t^i\}_{1\leq i\leq n}$ is a Cauchy sequence, indeed *first,* $\{(p(k_t^i)_{i\epsilon N}\}$ is bounded, thus exist and is stable, *second,* it can be extracted under convergent sequences from $\{k_t^i\}_{1\leq i\leq n}$ making $\{p(k_{\sigma(t)}{}^i)\}_{i\epsilon N}$ converge to the same limit as $\{(p(k_t^i)\}_{1\leq i\leq n}$. Therefore, in restricting the definition domain of financial assets to $B(0,\ M)$, then, $E(p(k_{\sigma(t)}{}^i))$ converge to a finite limit. If that limit is 0 and $Var(p(k_{\sigma(t)}{}^i))<\infty$, then volatility of asset prices is also low, thus yield to incentives to invest in financial assets stability and financial market improvement. Otherwise, if that limit tends toward ∞ because the supremum is unbounded i.e $M>Inf\{p(k_t^i))_{1\leq i\leq n}\}>1$, then, $\{(p(k_t^i)\}_{1\leq i\leq n}.\rightarrow\infty$ for a given $Var(p(k_{\sigma(t)}{}^i))$, then volatility of asset prices stability is difficult to establish since, the sequences, $\{(k_t^i)\}_{1\leq i\leq n}$. are not stationary, thus needs more elaborated treatment method such that, *cointegration method (*Engel and Granger*)*, to ensure stability econometrically

References

[1] Eicher, T., 1996, "Interaction between Endogenous Human Capital and Technological Change," *Review of Economic Studies*, 63, 127-144.

[2] Aghion, P., and P. Howitt, 1992, “A Model of Growth Through Creative Destruction”, *Econometrica,* 60, March, 323–351.

[3] Solow, R. M., 1956, “A Contribution to the Theory of Economic Growth”, *Quarterly Journal of Economics,* 70(1), 65-94.

[4] Denison, E. F., 1962, “The Sources of Economic Growth in the United States and the Alternatives before Us”, *Committee for Economic Development, New York.*

[5] Lucas, R. E., 1988, “On the Mechanics of Economic Development,” *Journal of Monetary Economics*, 22 (1), 3-42.

[6] Azariadis, C. and Drazen, A., 1990, “Threshold Externalities in Economic Development”, *Quarterly Journal of Economics,* 101, 501-526.

[7] Romer, P. M., 1986, “Increasing Returns and Long Run Growth,” *Journal of Political Economy*, 94 (5), 1002-1037.

[8] Romer, P. M. (1990), “Endogenous Technological Change,” *Journal of Political Economy*, 98 (5), 71-102.

[9] Loubaki, D., 2012, “On the Mechanics of the brain drain reduction in Poorest Developing Countries”, *Journal of Economic Development,* vol 37(3).

[10] Miao, J. and Wang, P., 2018, “Asset Bubbles and Credit Constraint,” *American Economic Review,* 108 (9).

[11] Tobin, J., 1969, “A General Equilibrium Approach to Monetary Theory”, *Journal of Money, Credit and Banking,* 1(1), 15-29.

[12] Tirole, J., 1982, “On the Possibility of Speculation under Rational Expectations”, *Econometrica,* 50, september

[13] Tirole, J., 1985, “Asset Bubbles and Overlapping Generations”, *Econometrica,* 53 (6), p. 1499-1528.

[14] Lucas, R., 1978, “Asset Prices in an Exchange Economy”, *Econometrica,* vol.46 (6), p. 1429-1445

[15] Grossman, G. M. and Yanagawa, N., 1992, “Asset Bubbles and Endogenous Growth”, *NBER Working Paper Series, N°4004.*