

On Non-Canonical Solving the Satisfiability Problem

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Abstract

We study the non-canonical method for solving the Satisfiability problem which given by a formula in the form of the conjunctive normal form. The essence of this method consists in counting the number of tuples of Boolean variables, on which at least one clause of the given formula is false. On this basis the solution of the problem obtains in the form YES or NO without constructing tuple, when the answer is YES. It is found that if the clause in the given formula has pairwise contrary literals, then the problem can be solved efficiently. However, when in the formula there are a long chain of clauses with pairwise a non-contrary literal, the solution leads to an exponential calculations.

Keywords

Satisfiability Problem, Satisfying Tuple, Disjunction, Clause, Conjunction, NP-Complete

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1. Introduction

To uniquely understand used terminology in the article, we recall some definitions.

Suppose we have a two-element set $E = \{0;1\}$. The function $f: E^n \rightarrow E$ is called Boolean. Every vector in E^n is considered as the tuple of values of variables x_1, x_2, \dots, x_n of the function $f(x_1, x_2, \dots, x_n)$.

A Boolean variable with negation or without negation is called literal. Literals x and \bar{x} is called contrary. The conjunction of r ($1 \leq r \leq n$) different non-contrary literals, called elementary. Any Boolean function can be represented as a disjunction of some elementary conjunctions. Such a representation of the function is called disjunctive normal form (DNF) of the function $f(x_1, x_2, \dots, x_n)$. For example, the function can be written as the following DNF:

$$f(x_1, x_2, x_3) = x_1 \vee x_2 x_3 \vee \bar{x}_2 \bar{x}_3$$

A disjunction of different non-contrary literals called clause. Any Boolean function can be represented as a conjunction of some clauses. For example, the function given above in the form of the DNF can be written as CNF:

$$f(x_1, x_2, x_3) = (x_1 \vee x_2 \vee \bar{x}_3)(x_1 \vee \bar{x}_2 \vee x_3).$$

In the theory of computational complexity, the problems, belonging to the class NP, are considered. The problem Z belongs to the class NP if its solution can be verified by the time, described as a polynomial function on the dimension of the problem. Usually, a problem of the class NP formulated as a decision problem, that is, in the form of a question to which the answer (solution) is YES or NO. Thus, the solution of the problem — it is the answer YES or NO [2]. To verify the correctness of the answer to this question, some certificate should be submitted that in polynomial time on the dimension of the problem verifies the correctness of the answer.

However, in practice, the solution of the problem in form of a set of some parameters of the problem (vertices, edges, literals) is used as a certificate, allowing form the answer

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YES or NO. To obtain the certificate in such form is often difficult [11].

Of particular importance in the theory of computational complexity plays the satisfiability problem (SAT), which is formulated as follows [1, 2]. Let it is given m clauses D_1, D_2, \dots, D_m , containing variables x_1, x_2, \dots, x_n . Has the formula $f(x_1, x_2, \dots, x_n) = D_1 \cdot D_2 \cdot \dots \cdot D_m$ satisfiable assignment?

The SAT problem is one of NP-complete problems [1-10]. Historically, this is the first NP-complete problem. Solution of this problem is important for theory and practice.

Currently, to solve the SAT problem, DPLL algorithms found the greatest application which are based on the works of Davis and Putnam [12] and Davis Lodzhmana and Loveland [13]. The essence of these algorithms is to find a satisfying assignment for the SAT problem. The complexity of these algorithms can be described by an exponential formula of the dimension of the problem, since it is related to the construction of the tree of values of the variables of the problem [14].

The solution of the Satisfiability problem is called non-canonical if the answer YES or NO was obtained without presentation of variable values when a given formula is true.

The purpose of this article is to investigate of non-canonical method for solving the Satisfiability problem.

2. Basis of the Method

Let the formula of the Satisfiability problem is:

$$f(x_1, x_2, \dots, x_n) = D_1 \cdot D_2 \cdot \dots \cdot D_m. \quad (1)$$

It is necessary to determine the existence of unknown values x_1, x_2, \dots, x_n such that satisfy the function (1).

The formula (1) can be transformed by the operation of negation on it. As a result, we will obtain a new formula:

$$\bar{f}(x_1, x_2, \dots, x_n) = C_1 \vee C_2 \vee \dots \vee C_m, \quad (2)$$

where each elementary conjunction obtained by the negation of the corresponding clause: $C_i = \bar{D}_i$ ($i = 1, 2, \dots, m$) in the formula (1).

Lemma 1. Each conjunction in the formula (2) determines a set of tuples of values of the variables of formula (1), for which the corresponding clause to be false.

Proof. In fact, in the formula (1), any clause is false if and only if every literal of this clause is false. Therefore, an

elementary conjunction of the formula (2) must be true for the tuples on which the corresponding clause is false. What we wanted to prove. \diamond

For example, suppose we have the formula:

$$f(x_1, x_2, x_3) = (x_1 \vee \bar{x}_2)(x_2 \vee \bar{x}_3)(\bar{x}_1 \vee x_3), \quad (3)$$

where $D_1 = x_1 \vee \bar{x}_2$, $D_2 = x_2 \vee \bar{x}_3$, $D_3 = \bar{x}_1 \vee x_3$.

The truth table for the clauses of this formula has the form:

Table 1. Table for the clauses.

x_1	x_2	x_3	D_1	D_2	D_3
0	0	0	1	1	1
0	0	1	1	0	1
0	1	0	0	1	1
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	1	0	1
1	1	0	1	1	0
1	1	1	1	1	1

Executing a negation operation on the formula (3), we obtain the formula:

$$\bar{f}(x_1, x_2, x_3) = \bar{x}_1 \cdot x_2 \vee \bar{x}_2 \cdot x_3 \vee x_1 \cdot \bar{x}_3 \quad (4)$$

Thus, we conclude that the set of elementary conjunctions of the formula (2) represents the set of all tuples of variables x_1, x_2, \dots, x_n , on which at least one clause of formula (1) is false.

Theorem 1. The Satisfiable problem, determined on a set of variables, has a satisfiable assignment if and only if the total number of different tuples, on which at least one clause in the formula (1) is false, strictly smaller than 2^n .

Proof. This is evident. If the total number of tuples, in which at least one clause of formulas (1) is false, is equal to 2^n , then the satisfiable assignment does not exist for the Satisfiability problem. \diamond

3. Certificate

Thus, according to Theorem 1 if we calculate the total number of different tuples, in which at least one clause in the Satisfiability problem is false, then on this basis, we can formulate the answer YES or NO to the question about existence of the satisfying assignment, that is, one can solve the Satisfiability problem, without constructing satisfying assignment. In this case, the certificate counts the total number of different tuples, in which at least one clause of the formula (1) is false.

Consider the different cases of the counting of tuples, in which clause of the formula (1) is false.

Lemma 2. Each conjunction C_i ($i=1,2,\dots,m$) of the equation (2), containing r_i ($1 \leq r_i \leq n$) literals, determines 2^{n-r_i} tuples on which the corresponding clause of the formula (1) is false.

Proof. Let us consider the elementary conjunction in the formula (2). Let C_i ($i=1,2,\dots,m$) is one of them. If the number of literals that make up the C_i is r_i ($1 \leq r_i \leq n$), then this conjunction defines 2^{n-r_i} of tuples on which the corresponding clause of the formula (1) is false, because the $(n - r_i)$ missing literals can take all possible values. \diamond

So, in the example (see Table 1 above), conjunction $C_1 = \bar{x}_1 \cdot x_2$ determines $2^{3-2} = 2$ tuples on which clause $D_1 = x_1 \vee \bar{x}_2$ is false. Here we have $n = 3$; $r_i = 2$. These are tuples (010) and (011).

Lemma 3. Any pair of clauses C_i and C_k ($i, k=1,2,\dots,m; i \neq k$) of formula (2), containing contrary literals together, determines $2^{n-r_i} + 2^{n-r_k}$ of tuples on which the corresponding clause of the formula (1) is false.

Proof. Let us now consider two clauses of the formula (2). Let it be the conjunctions of the C_i and C_k ($i, k=1,2,\dots,m; i \neq k$). Obviously, if the literal of the C_i has the contrary literal in C_k , then these conjunctions determine different tuples of variables. \diamond

For example, two elementary conjunctions in the formula (4) $C_1 = \bar{x}_1 \cdot x_2$ and $C_3 = x_1 \cdot \bar{x}_3$ ($n = 3$) contain the variable x_1 with negation and without negation. The conjunction C_3 determines tuples (100) and (110), which does not coincide with the tuples, represented by the conjunction C_1 ((010), (011)).

The considered conjunctions are called alternative.

Theorem 2. Let the formula (2) contains only pairwise alternative conjunctions C_i, C_k ($i, k=1,2,\dots,m; i \neq k$). Then the total number of tuples, on which the corresponding clause of the formula (1) is false, is equal

$$\sum_{i=1}^m 2^{n-r_i}.$$

Proof. It follows from Lemma 3. \diamond

It is easy to see that the formula (4) contains pairwise alternative conjunctions. The total number of tuples, on which the corresponding clauses in the formulas (3) is false, is equal:

$$2^1 + 2^1 + 2^1 = 6,$$

that is, the formula (3) has $2^3 - 6 = 2$ satisfiable tuples. In this case, the solution of the problem is YES.

Theorem 3. Any Boolean function, represented by the formula (2), can be transformed into a formula containing only pairwise alternative conjunctions.

Proof. Any Boolean function can be represented as a perfect disjunctive normal form (PDNF), which contains only pairwise alternative conjunctions, each of which depends on n variables. For example, a pair of conjunctions $x_1 x_2 \vee x_1 x_3$ can be transformed as follows:

$$\begin{aligned} x_1 x_2 \vee x_1 x_3 &= x_1 x_2 (x_3 \vee \bar{x}_3) \vee x_1 (x_2 \vee \bar{x}_2) x_3 = \\ &= x_1 x_2 x_3 \vee x_1 x_2 \bar{x}_3 \vee x_1 \bar{x}_2 x_3. \diamond \end{aligned}$$

Unfortunately the time (the number of operations), that is required to build PDNF, can be unacceptably large.

Let there is some conjunction C_i containing r_i literals from n ($r_i \leq n$). The variables, which do not have their own literals in C_i , called *free* for the given conjunction. The number of such free variables in the conjunction C_i equal to $t_i = n - r_i$. Similarly, if there are p conjunctions C_1, C_2, \dots, C_p , then the total number of free variables in these conjunctions is designated as $t_{1,2,\dots,p}$.

Theorem 4. Let the formula (2) has p conjunctions C_1, C_2, \dots, C_p , having no pairwise alternative literals. Then the total number of tuples, on which this conjunction is false, is equal to:

$$R = \sum_{i=1}^p 2^{t_i} - \sum_{\forall i < j} 2^{t_{i,j}} + \sum_{\forall i < j < k} 2^{t_{i,j,k}} - \dots + (-1)^{p-1} 2^{t_{1,2,\dots,p}}. \quad (5)$$

Proof. The validity of Theorem 4 follows from the method of inclusions and exclusions. \diamond

Consider a chain of pairwise non-alternative conjunctions $C_1 = x_1, C_2 = x_2, C_3 = x_3 \cdot x_4$. The total number of tuples, on which the corresponding disjunctions are false, is equal to:

$$R = 2^3 + 2^3 + 2^2 - 2^2 - 2^1 - 2^1 + 2^0 = 13.$$

Theorem 5. Let the formula (2) has p conjunctions C_1, C_2, \dots, C_p , having no pairwise alternative literals. Then the number of members in the formula (5) is equal to 2^{p-1} .

Proof. The number of members in the formula (5) equals the number of subsets of a finite p -set without empty subset. \diamond

4. Conclusions

Consideration of non-canonical method for solving the Satisfiability problem shows that this method can be used effectively for solving problems, initially containing pairwise of alternative conjunctions. Unfortunately, in the general case, solving the problem by the considered method is ineffectively as it involves performing an exponential number of operations. Therefore, this method can be used with a limited number of chains of pairwise non-alternative conjunctions. Obviously, this depends on the performance of the used computer.

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