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Analytical Solution of Nonlinear Dynamical System Based on Homotopy Pade Approximate

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Abstract

In this paper, the homotopy Pade technique is presented as an alternative method to derive the analytical solution for nonlinear dynamical system. Illustrative example is used to show the validity and accuracy of the method in solving the nonlinear system. Comparisons are conducted between the analytical approximation and numerical solution. The results obtained here demonstrate that the homotopy Pade approximate is an effective and robust technique for nonlinear dynamical systems.

Keywords

Homotopy Analysis Method, Homotopy Pade Approximate, Nonlinear Dynamical System

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1. Introduction

Nonlinear dynamical systems are omnipresent in numerous practical engineering and mathematics problems. It is hardly to seek the exact solutions in normal circumstances. However, the development of analytical methods can provide an all-embracing understanding for the systems. The homotopy analysis method (HAM) [1] is a robust analytical approximate technique for solving a class of nonlinear problems. The originated idea of the HAM was enlightened from the homotopy in topology. Inasmuch as the accuracy and validity of the HAM is not depended on the presence of small parameters in the governing systems of motion, it overcomes the foregoing restrictions of conventional analysis methods.

For more than two decade, a number of scholars have adopted the HAM to a variety of nonlinear problems in mathematics and engineering [2-12]. In this paper, the application of the HAM is exploited to nonlinear dynamical systems. The significance of dynamical systems is mainly due to its global bifurcation, regular and chaotic motions, the intensive research subjects are thus at the forefront of nonlinear dynamics. Recently, some achievements and fruitful outcome have been established for dynamical systems [13-18]. Some optimal HAM approaches are developed, which can get faster convergent homotopy series solution [19-27].

The objective of the present work is to conduct a quantitative analysis for nonlinear dynamical systems. The example is selected to substantiate the validity and accuracy of the homotopy Pade approximate technique. Comparisons are carried out between the results of these analytical method and the exact solutions. Because the selected example herein are exactly solvable, therefore its results demonstrate that the homotopy Pade approximate solutions are highly accurate for nonlinear dynamical system.

2. Homotopy Pade Approximate

Pade approximate expands a function as a ratio of two power series. If the rational function is

$$R(t) = \frac{\sum_{k=0}^{M} a_k t^k}{1+\sum_{k=1}^{N} b_k t^k}, \tag{1}$$

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then $R(t)$ is said to be a $[M,N]$ Pade approximate of the series

$$f(t)=\sum_{k=0}^{\infty}c_k t^k, \tag{2}$$

and

$$R(0)=f(0),\ \frac{d^k}{dt^k}R(t)\big|_{t=0}=\frac{d^k}{dt^k}f(t)\big|_{t=0},$$
$$(k=1,2,\cdots,M+N). \tag{3}$$

Equation (3) provides $M+N+1$ equations for the unknowns parameters $a_0,\cdots,a_M$ and $b_1,\cdots,b_N$. The homotopy Pade approximate technique is a combination of the above mentioned traditional Pade technique with the HAM.

3. Illustrative Example and Discussion

Consider the nonlinear system

$$\frac{d^2x_1}{dt^2}+x_2\frac{dx_1}{dt}-x_1=1, \tag{4a}$$

$$\frac{d^2x_2}{dt^2}-x_1\frac{dx_2}{dt}-x_2=1, \tag{4b}$$

with the initial conditions

$$x_1(0)=1,\ x_1'(0)=1,\ x_2(0)=1,\ x_2'(0)=-1. \tag{5}$$

The exact solutions for Eqs. (4) subject to the initial conditions in Eq. (5) are

$$x_1(t)=\mathrm{e}^t,\ x_2(t)=\mathrm{e}^{-t}. \tag{6}$$

Suppose that the solution can be expressed by a set of base functions $\{t^n \mid n=0,1,2,3,\cdots\}$, we choose the initial approximation as

$$x_{1,0}(t)=1+t,\ x_{2,0}(t)=1-t. \tag{7}$$

Defining the nonlinear operator as

$$N\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\end{pmatrix}=\begin{pmatrix}\dfrac{\partial^2\varphi_1(t;q)}{\partial t^2}+\varphi_2(t;q)\dfrac{\partial\varphi_1(t;q)}{\partial t}-\varphi_1(t;q)\\ \dfrac{\partial^2\varphi_2(t;q)}{\partial t^2}-\varphi_1(t;q)\dfrac{\partial\varphi_2(t;q)}{\partial t}-\varphi_2(t;q)\end{pmatrix} \tag{8}$$

Thus, the zeroth-order deformation equation can be written in the form

$$(1-q)L\begin{pmatrix}\varphi_1(t;q)-x_{1,0}(t)\\ \varphi_2(t;q)-x_{2,0}(t)\end{pmatrix}=q\hbar\left[N\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}\right], \tag{9}$$

And the mth-order deformation equation can be expressed as

$$L\begin{pmatrix}x_{1,m}(t)-\chi_m x_{1,m-1}(t)\\ x_{2,m}(t)-\chi_m x_{2,m-1}(t)\end{pmatrix}=\hbar\begin{pmatrix}R_{1,m}(\mathrm{x}_{1,m-1})\\ R_{2,m}(\mathrm{x}_{2,m-1})\end{pmatrix}. \tag{10}$$

From the initial conditions and the initial approximation, we have

$$x_{1,m}(0)=0,\ x_{2,m}(0)=0,\ x_{1,m}'(0)=0,$$
$$x_{2,m}'(0)=0,\ (m\ge 1) \tag{11}$$

and

$$R_{1,m}(\mathrm{x}_{1,m-1})=x_{1,m-1}''(t)-x_{1,m-1}(t)+\sum_{i+j=m-1}x_{1,i}'(t)x_{2,j}(t)-(1-\chi_m), \tag{12a}$$

$$R_{2,m}(\mathrm{x}_{2,m-1})=x_{2,m-1}''(t)-x_{2,m-1}(t)-\sum_{i+j=m-1}x_{2,i}'(t)x_{1,j}(t)-(1-\chi_m). \tag{12b}$$

From Eq. (10), it implies that

$$x_{1,m}(t)=\chi_m x_{1,m-1}(t)+\hbar\int_0^t\left(\int_0^\tau R_{1,m}(\mathrm{x}_{1,m-1})\mathrm{d}s\right)\mathrm{d}\tau+C_{1,m}t+C_{2,m}, \tag{13a}$$

$$x_{2,m}(t)=\chi_m x_{2,m-1}(t)+\hbar\int_0^t\left(\int_0^\tau R_{2,m}(\mathrm{x}_{2,m-1})\mathrm{d}s\right)\mathrm{d}\tau+C_{3,m}t+C_{4,m}, \tag{13b}$$

$$R_{1,1}(\mathrm{x}_{1,0})=x_{1,0}''(t)-x_{1,0}(t)+x_{1,0}'(t)x_{2,0}(t)-1=-t-1+1-t-1=-2t-1 \tag{14a}$$

$$R_{2,1}(\mathrm{x}_{2,0})=x_{2,0}''(t)-x_{2,0}(t)-x_{2,0}'(t)x_{1,0}(t)-1=t-1+1+t-1=2t-1, \tag{14b}$$

$$x_{1,1}(t)=-\hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right), \tag{15a}$$

$$x_{2,1}(t)=\hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right), \tag{15b}$$

$$R_{1,2}(\mathrm{x}_{1,1}) = x''_{1,1}(t) - x_{1,1}(t) + x'_{1,0}(t)x_{2,1}(t) + x'_{1,1}(t)x_{2,0}(t)$$

$$= \hbar(-2t-1) + \hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right) + \hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right) - \hbar\left(t^2+t\right)(1-t)$$

$$= \hbar\left(\frac{5t^3}{3} - 3t - 1\right), \tag{16a}$$

$$R_{2,2}(\mathrm{x}_{2,1}) = x''_{2,1}(t) - x_{2,1}(t) - x'_{2,0}(t)x_{1,1}(t) - x'_{2,1}(t)x_{1,0}(t)$$

$$= \hbar(2t-1) - \hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right) - \hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right) - \hbar\left(t^2-t\right)(1+t)$$

$$= \hbar\left(-\frac{5t^3}{3} + 3t - 1\right), \tag{16b}$$

$$x_{1,2}(t) = -\hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right) + \hbar^2\left(-\frac{t^2}{2}-\frac{t^3}{2}+\frac{t^5}{12}\right), \tag{17a}$$

$$x_{2,2}(t) = \hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right) + \hbar^2\left(-\frac{t^2}{2}+\frac{t^3}{2}-\frac{t^5}{12}\right). \tag{17b}$$

We now successively obtain the second-order analytical approximation by HAM as the following

$$x_1(t) \approx 1 + t - \hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right) - \hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right) + \hbar^2\left(-\frac{t^2}{2}-\frac{t^3}{2}+\frac{t^5}{12}\right)$$

$$= 1 + t - \left(\hbar + \frac{\hbar^2}{2}\right)t^2 - \left(\frac{2\hbar}{3}+\frac{\hbar^2}{2}\right)t^3 + \frac{\hbar^2}{12}t^5 \lim_{x\to\infty} \tag{18a}$$

$$x_2(t) \approx 1 - t + \hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right) + \hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right) + \hbar^2\left(-\frac{t^2}{2}+\frac{t^3}{2}-\frac{t^5}{12}\right)$$

$$= 1 - t - \left(\hbar + \frac{\hbar^2}{2}\right)t^2 + \left(\frac{2\hbar}{3}+\frac{\hbar^2}{2}\right)t^3 - \frac{\hbar^2}{12}t^5 \tag{18b}$$

The valid region of $\hbar$ can be determined from the $\hbar-$curve in Fig. 1. The series of $x_1(t)$ and $x_2(t)$ converges at $-1.3<\hbar<-0.6$. Let $\hbar=-1$, the 10th-order analytical approximation series solutions of the HAM are

$$\begin{aligned} x_1(t) = {} & 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040} + \frac{t^8}{40320} + \frac{t^9}{362880} + \frac{t^{10}}{3628800} \\ & + \frac{t^{11}}{39916800} + \frac{t^{13}}{47900160} - \frac{t^{14}}{10897286400} - \frac{t^{15}}{108972864000} + \frac{29119t^{16}}{46702656000} \\ & - \frac{10886489t^{17}}{12703122432000} + \frac{909977417t^{18}}{533531142144000} - \frac{25070839867t^{19}}{5529322745856000} \\ & - \frac{153073227809t^{20}}{76028187755520000} + \frac{2253389787941t^{21}}{1021818843434188800} \end{aligned} \tag{19a}$$

$$\begin{aligned} x_2(t) = {} & 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} + \frac{t^6}{720} - \frac{t^7}{5040} + \frac{t^8}{40320} - \frac{t^9}{362880} + \frac{t^{10}}{3628800} \\ & - \frac{t^{11}}{39916800} - \frac{t^{13}}{47900160} - \frac{t^{14}}{10897286400} + \frac{t^{15}}{108972864000} + \frac{29119t^{16}}{46702656000} \\ & + \frac{10886489t^{17}}{12703122432000} + \frac{909977417t^{18}}{533531142144000} - \frac{25070839867t^{19}}{5529322745856000} \\ & + \frac{10886489t^{17}}{12703122432000} + \frac{909977417t^{18}}{533531142144000} - \frac{25070839867t^{19}}{5529322745856000} \end{aligned} \tag{19b}$$

The $[5,5]$ homotopy Pade approximate solutions of $x_1(t)$ and $x_2(t)$ are

$$x_1(t) = \frac{1 + \frac{t}{2} + \frac{t^2}{9} + \frac{t^3}{72} + \frac{t^4}{1008} + \frac{t^5}{30240}}{1 - \frac{t}{2} + \frac{t^2}{9} - \frac{t^3}{72} + \frac{t^4}{1008} - \frac{t^5}{30240}} \tag{20a}$$

$$x_2(t)=\frac{1-\dfrac{t}{2}+\dfrac{t^2}{9}-\dfrac{t^3}{72}+\dfrac{t^4}{1008}-\dfrac{t^5}{30240}}{1+\dfrac{t}{2}+\dfrac{t^2}{9}+\dfrac{t^3}{72}+\dfrac{t^4}{1008}+\dfrac{t^5}{30240}}. \tag{20b}$$

A comparison is shown between the 10th-order homotopy analysis approximate solutions and exact solutions for $x_1(t)$ and $x_2(t)$ in Fig. 2. From Fig. 3 we can see that the [5, 5] homotopy Pade approximate solutions provides excellent agreement with the exact solutions. The algorithm is coded by the symbolic computation software Mathematica.

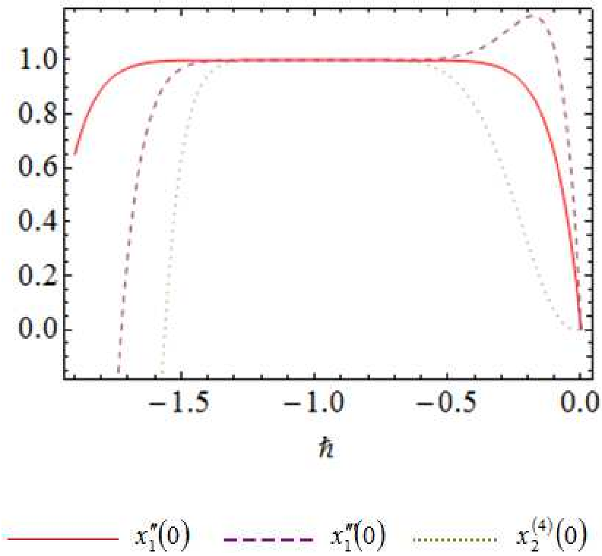

Figure 1. $\hbar-$ curves of $x_1''(0)$, $x_1'''(0)$ and $x_2^{(4)}(0)$ obtained from 10th-order approximation for Eq. (4).

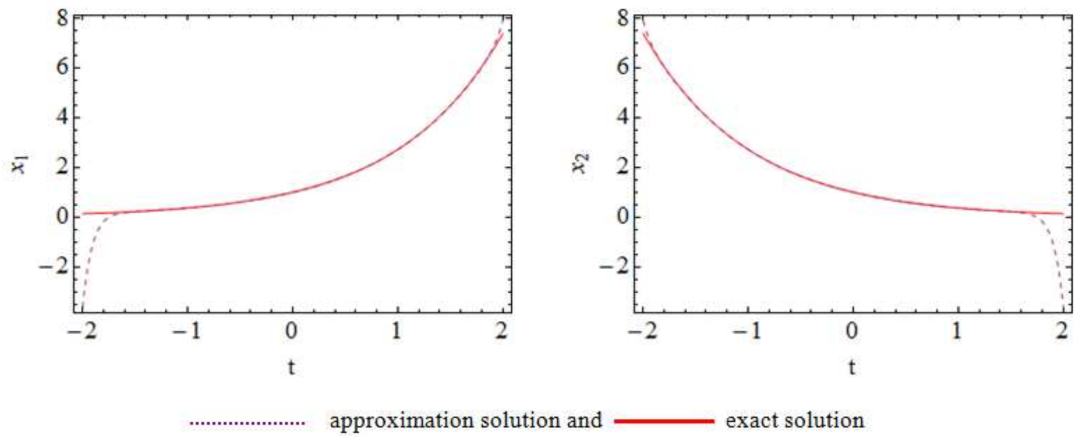

Figure 2. Comparison of 10th-order homotopy analysis approximation and exact solution for $x_1(t)$ and $x_2(t)$.

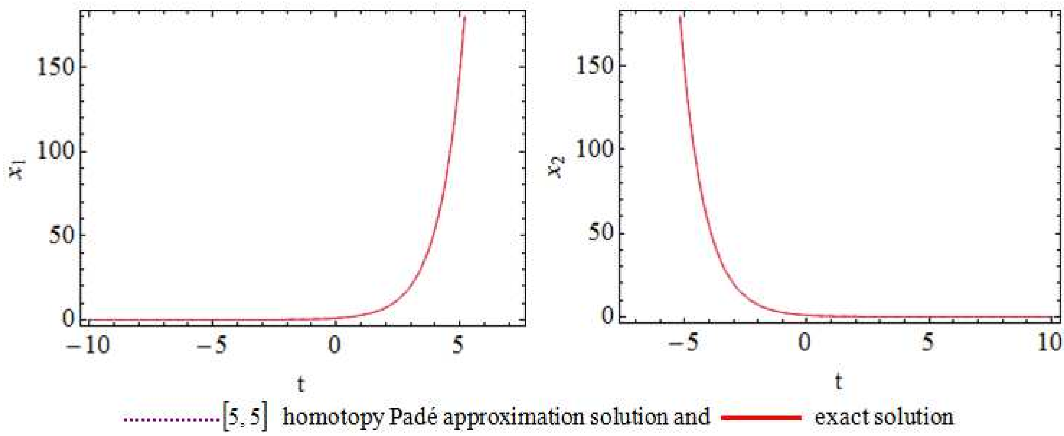

Figure 3. Comparison of [5, 5] homotopy Pade approximation and exact solution for $x_1(t)$ and $x_2(t)$.

4. Conclusions

In summary, homotopy Pade approximate is applied to obtain analytical approximation solution for dynamical system. The fundamental idea of the method is essentially different from other existing analytical methods. The homotopy Pade approximate provide an ingenious avenue for controlling the convergences of approximation series. The exact solutions of this example can be used to verify the accuracy of the method. By solving such example, it is illustrated that the present technique are not an ad-hoc approach, it can be generalized to investigate time-variant and time-invariant systems. Because of its flexibility, the present techniques can also be further generalized to investigate more complicated nonlinear dynamical systems that can only be solved by numerical approaches.

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