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Polynomial-time Algorithm for Determining the Graph Isomorphism

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Abstract

The methodology of positioning graph vertices relative to each other to solve the problem of determining isomorphism of two undirected graphs is developed. Based on the position of the vertex in one of the graphs, it is determined the corresponding vertex in the other graph. For the selected vertex of the undirected graph, define the neighborhoods of the vertices are defined. Next, it is constructed the auxiliary directed graph, spawned by the selected vertex. The vertices of the digraph are positioned by special characteristics — vectors, which locate each vertex of the digraph relative the found neighborhoods. This enabled to develop the polynomial-time algorithm for determining graph isomorphism.

Keywords

Isomorphism, Algorithm, Graph, Graph Isomorphism Problem

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1. Introduction

Let L_n is the set of all n-vertex undirected graphs without loops and multiple edges.

Let, further, there is a graph $= (V_G, E_G) \in L_n$, where $V_G = (v_1, v_2, \dots, v_n\}$ is the set of graph vertices and $E_G = \{e_1, e_2, \dots, e_m\}$ is the set of graph edges. Local degree $\deg(v)$ of a vertex $v \in V_G$ is the number of edges which incident to the vertex v. Every graph $G \in L_n$ can be characterized by the vector $D_G = (\deg(v_{i_1}), \deg(v_{i_2}, \dots, \deg(v_{i_n}))$ of local vertex degrees, where $\deg(v_i) \leq \deg(v_j)$, if $i < j$.

The graphs $G = (V_G, E_G)$, $H = (V_H, E_H) \in L_n$ are called isomorphic if between their vertices there exists an one-to-one (bijective) correspondence $\varphi : V_G \leftrightarrow V_H$ such that if $e_G = \{v, u\} \in E_G$ then the corresponding edge $e_H = \{\varphi(v), \varphi(u)\} \in E_H$, and conversely [1, 2]. The graph isomorphic problem consists in determining isomorphism graphs $G, H \in L_n$.

The problem of determining the isomorphism of two given undirected graphs is used to solve chemical problems, and to optimize programs [3-6]. Effective (polynomial-time) algorithms for solving this problem were found for some narrow classes of graphs [7-9]. However for the general case, effective methods for determining the isomorphism of graphs are not known [10].

The purpose of this article is to propose the polynomial-time algorithm for determining isomorphism of the undirected graphs.

2. Algorithm Bases

The methodology of positioning graph vertices relative to each other to solve the problem of determining isomorphism of two undirected graphs is developed. Based on the position of the vertex in one of the graphs, it is determined the corresponding vertex in the other graph.

Consider the elements of the algorithm.

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Let there be a graph $G \in L_n$. For the vertex $v \in V$ of the graph G we define the concept of neighborhood of k-th level ($0 \le k \le n-1$).

The neighborhood of 1-st level of the vertex v is the set of all graph vertices that are adjacent to the vertex v. In general, a neighborhood of k-th level is the set of all graph vertices that are adjacent to the vertices (k−1)-th level. Such the neighborhood is denoted by $N_G^{(k)}(v)$. For convenience, it is assumed that the vertex v forms the neighborhood of the zero level.

For the given vertex v by means of the graph G an auxiliary directed graph $\vec{G}(v)$ is constructed, spawned by the vertex $v \in V$, as follows.

The set of vertices belonging to one and the same neighborhood of the vertex v, form a line of graph vertices. Each line has the same number as the level of a neighborhood of the vertex v. Further, if in the initial graph G the edge $\{v, u\}$ connects the vertex v belonging to lines with lower number than vertex u then such an edge to replace by the arc (v, u), outgoing from the vertex v and incoming to the vertex u. If the edge of the graph connects vertices of one and the same line, this edge is replaced by two opposite arcs.

Figure 1 presents the graph $G = (V, E) \in L_n$ and the auxiliary digraph $\vec{G}(v_1)$, spawned according to this graph by the vertex $v_1 \in V$.

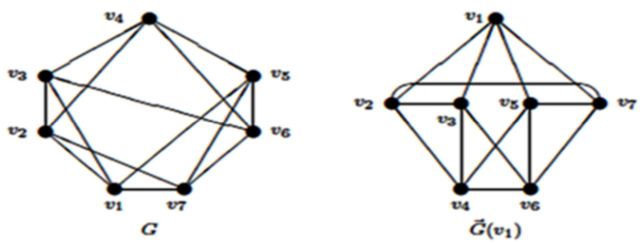

Figure 1. Graph G and auxiliary digraph $\vec{G}(v_1)$.

Here $N_G^{(0)}(v_1) = \{v_1\}$ is its own neighborhood (line) of the 0-th level of the vertex v_1, the set of vertices $N_G^{(1)}(v_1) = \{v_2, v_3, v_6, v_7\}$ forms a neighborhood (line) of 1-st level and, finally, the set $N_G^{(2)}(v_1) = \{v_3, v_6\}$ is the neighborhood of the 2-nd level of the vertex v_1.

Each vertex v of the auxiliary digraph $\vec{G}(v)$ is characterized by two vectors I_v and O_v.

Elements of vectors I_v and O_v are numbers. These numbers are the line numbers of vertices of the auxiliary digraph. Vector I_v contains the line numbers of vertices, from which arcs of the digraph come into the vertex v, and the vector O_v contains the line numbers of vertices that receive arcs from vertex v. If several arcs income into the vertex v from one and the same line of digraph then the line number is repeated in the vector accordingly. Similarly, if several arcs from the vertex v come into one and the same line, this number is also repeated in the vector accordingly. The elements of the vectors I_v and O_v assume ordered in ascending order of value of numbers.

The vectors I_v and O_v will be called the characteristics of the vertex v, and I_v is input and O_v is output characteristics. Characteristics of the two vertices v_1, v_2 are equal, if their input and output characteristics are equal, respectively, i.e., if $I_{v_1} = I_{v_2}$ and $O_{v_1} = O_{v_2}$.

Find characteristics of the vertices of the auxiliary digraph $\vec{G}(v_1)$, shown in Figure 1 (the arcs of this digraph are directed from top to bottom, and the horizontal edges represent two oppositely directed arcs). Results are presented on Table 1.

Table 1. The vertex characteristics of the digraph $\vec{G}(v_1)$.

$I_{v1} = ø$;	$I_{v2} = (0,1,1)$;	$I_{v3} = (0,1)$;	$I_{v4} = (1,1,1,2)$;
$O_{v1} = (1,1,1,1)$;	$O_{v2} = (1,1,2)$;	$O_{v3} = (1,2,2)$;	$O_{v4} = (2)$;
$I_{v5} = (0,1)$;	$I_{v6} = (1,1,1,2)$;	$I_{v7} = (0,1,1)$;	
$O_{v5} = (1,2,2)$;	$O_{v6} = (2)$;	$O_{v7} = (1,1,2)$.	

Auxiliary digraphs $\vec{G}(v)$ and $\vec{H}(u)$ is called positionally equivalent if the lines of graphs of the same level have the equal number of vertices, respectively, having equal characteristics.

In general, the positionally equivalent digraphs have arcs connecting the vertices of the same level. This introduces an element of equivocation, i.e. in this case, it cannot say that the positionally equivalent digraphs determine isomorphic graphs G and H.

A vertex $v \in V_G$ called unique in digraph $\vec{G}(v)$ if doesn't exist other vertex with the characteristics equal to characteristics of the vertex v.

It is easy to see that the vertex v_1 is the unique vertex of the constructed digraph (see Table 1).

Theorem 1. Let the graphs G and H are isomorphic. Let, further, the auxiliary digraphs $\vec{G}(v)$ and $\vec{H}(u)$ are the positionally equivalent, and each of digraphs has an unique vertex $v_i \in V_G$ and $u_j \in V_H$ such that $I_{v_i} = I_{u_j}$, $O_{v_i} = O_{u_j}$. Then between the vertices of digraphs there exists a bijective correspondence φ such that $u_j = \varphi(v_i)$.

Proof. Let the conditions of Theorem 1 are satisfied. As the graphs G and H are isomorphic then between vertices of positionally equivalent digraphs $\vec{G}(v_i)$ and $\vec{H}(u_j)$, having equal characteristics, there is a bijective correspondence φ. Since the vertices v_i and u_j have the same unique positioning in these digraphs then the relation $u_j = \varphi(v_i)$ is performed in any correspondence φ. In particular, that sort unique vertices are always vertices which spawn the positionally equivalent auxiliary digraphs.

Note that in general case, the vertices of graphs *G* and *H* having the same local degree, spawn different auxiliary digraphs.

Theorem 2. Suppose there are two isomorphic graphs $G = (V_G, E_G)$, $H = (V_H, E_H) \in L_n$. Suppose, further, for the subgraph $G_1 \subseteq G$, induced by the vertex set $X \subseteq V_G$, the auxiliary directed graph $\vec{G}_1(x)$ $(x \in X)$ has constructed. Then, in the graph *H*, there exists a subgraph $H_1 \subseteq H$, spawned by the set of vertices $Y \subseteq V_H$, such that it has auxiliary digraph $\vec{H}_1(y)$ $(y \in Y)$ that is positionally equivalent to the digraph $\vec{G}_1(x)$.

Proof. The validity of the above statement follows from the definition of isomorphism of graphs, and the same sequence of construction of any of the auxiliary digraphs.

3. Algorithm

Using the results of the previous section, it can proposes several algorithms for determining the isomorphism of two given graphs $G, H \in L_n$. The simplest of them is as follows.

To determine the fact of isomorphism of graphs *G* and *H*, it neseccary look for equally positioned vertex graphs.

Find the vertices $v \in V_G$ and $u \in V_H$ having positionally equivalent auxiliary digraphs in the graphs *G* and *H*. Remove the found vertices from the graphs together with the incident edges and repeat the process until we have exhausted the list of vertices of graphs. If at some point in the graphs *G* and *H* cannot find a pair of vertices having positionally equivalent auxiliary digraphs, then stop the calculation, since the graphs are not isomorphic.

Next, the proposed algorithm is described in more detail.

Input of the algorithm: graphs $G = (V_G, E_G), H = (V_H, E_H) \in L_n$, isomorphism of which must be determined, if it exists. It is believed that these graphs have the same number of vertices and edges, as well as their vectors of local degrees D_G and D_H are equal.

Output of the algorithm: conclusion about the isomorphism of the graphs *G* and *H*.

Algorithm for determining isomorphism graphs.

Step 1. Put $Q = G$, $S = H$, $N_G := n$, $N_H := n$, $i := 1$, $j := 1$.

Step 2. Choose the vertex $v_i \in V_Q$ in the graph *Q*.

Step 3. Construct the auxiliary digraph $\vec{Q}(v_i)$ using the graph *Q*.

Step 4. Find the characteristics of vertices of the auxiliary digraph $\vec{Q}(v_i)$.

Step 5. Choose the vertex $u_j \in V_S$ in the graph *S*.

Step 6. Construct the auxiliary digraph $\vec{S}(u_j)$ using the graph *S*.

Step 7. Find the characteristics of vertices of the auxiliary digraph $\vec{S}(u_j)$.

Step 8. Compare the characteristics of the vertices of the digraphs $\vec{Q}(v_i)$ and $\vec{S}(u_j)$ in the neighborhoods of the vertices v_i and u_j of the same level.

Step 9. If the digraphs $\vec{Q}(v_i)$ and $\vec{S}(u_j)$ are positionally equivalent then put $V_Q := V_Q \setminus \{v_i\}$, $V_S := V_S \setminus \{u_j\}$, $N_G := N_G - 1$, $N_H := N_H - 1$. Go to Step 11.

Step 10. If $N_H \neq 0$ then put $j := j + 1$, $N_H := N_H - 1$ and go to Step 5. Otherwise finish the computing as the graphs *G* and *H* are not isomorphic.

Step 11. If $N_G \neq 0$ then put $i := i + 1$, $j := 1$ and go to Step 2. Otherwise, stop the computing. The graphs G and *H* are isomorphic.

Theorem 3. The algorithm for determining the graph isomorphism determines an isomorphism of the given graphs if it exists.

Proof. The Theorem 1 is established that if graphs *G* and *H* are isomorphic then the pair of the unique vertices *v* and u, that spawn positionally equivalent digraphs $\vec{G}(v)$ and $\vec{H}(u)$, belong to some bijective mapping φ of the vertices of this graph. Therefore, removal of vertices *v* and *u* of graphs *G* and *H* together with incident edges, leads to the construction of subgraphs of $G^* \subset G$ and $H^* \subset H$, which is also isomorphic. Therefore, the repetition of the above procedure will lead to the exhaustion of the list of vertices isomorphic graphs *G* and *H*.

Now suppose that the graphs *G* and *H* are not isomorphic and the vertex $v \in V_G$, $u \in V_H$ spawn positionally equivalent digraphs $\vec{G}(v)$ and $\vec{H}(u)$. Then, obviously, in these digraphs there exist some subsets of vertices $X \subset V_G$ and $Y \subset V_H$, having corresponding equal characteristics, which spawn different (non-isomorphic) subgraphs. Here $v \in (V_G \setminus X)$, $u \in (V_H \setminus Y)$ and $Card(X) = Card(Y)$.

Removing sequentially from graph *Q* and *S* vertices, spawning the equivalent auxiliary digraphs, we obtain non-isomorphic subgraphs. As a result, the algorithm terminates without possibility finding a pair of vertices with equivalent digraphs.

Unfortunately, the collection of pairs of vertices, determined by the proposed algorithm, as a whole, does not always determine the bijective mapping of the vertices in initial isomorphic graphs. This is the price for the simplicity of the algorithm.

Theorem 4 The running time of the algorithm for determining the graph isomorphism equal to $O(n^4)$.

Proof. Determine the running time of the algorithm in steps 5–10. Steps 5, 10 need to spend one time unit at each step.

Steps 6–8 require to spend $O(n^2)$ time units each. Steps 9 require to spend $O(n)$ time units.

Therefore, n-multiple executing steps 5–10 requires to spend $O(n^3)$ time units.

Executing Steps 2–10 one times requires to spend $O(n^3)$ time units and executing n times requires $O(n^4)$ time units.

4. Conclusion

The efficient (polynomial-time) algorithms for determining isomorphism of undirected graphs was developed. With that end in view, it is used the methodology of positioning vertices concerning the selected vertex and its neighborhoods as the input and output characteristics of vertices. It allowed effectively to compare the structure of the given graphs.

Apparently, ideology of the positioning vertices, developed in this study, can be used in solving the problem of finding a subgraph which is isomorphic to the given graph. Although in this case we should expect origin of essential search.

Appendix

The follow example illustrates the work of the proposed algorithm.

Let two graphs G, $H \in L_n$ are given, for which it is necessary to determine the isomorphism (see Figure A1).

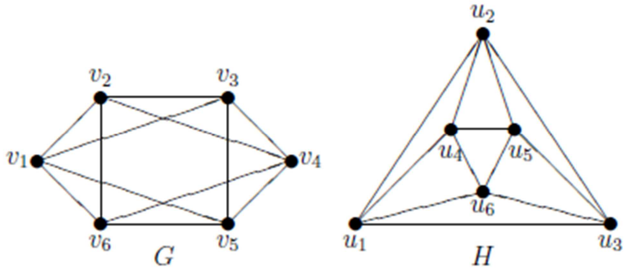

Figure A1. The initial graphs *G* and *H*.

The given graphs have the equal number of vertices, edges and vectors of degrees.

The vertex v_1 is choosen in the graph G and the auxiliary digraph $\vec{G}(v_1)$ is constructed (see Figure A2).

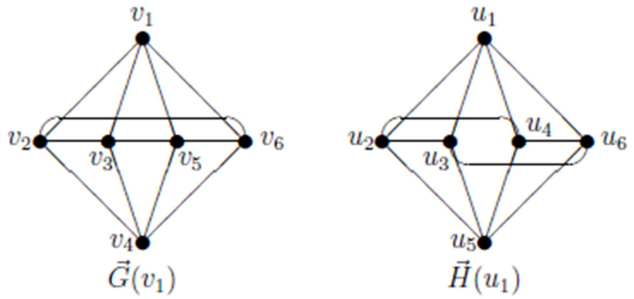

Figure A2. The auxiliary digraphs $\vec{G}(v_1)$ and $\vec{H}(u_1)$.

The computed characteristics of vertices of the constructed digraph are given in the Table A1.

Table A1. The vertex characteristics of the digraph $\vec{G}(v_1)$.

I_{v1} = ø;	I_{v2} = (0,1,1);	I_{v3} = (0,1,1);
O_{v1} = (1,1,1,1);	O_{v2} = (1,1,2);	O_{v3} = (1,1,2);
I_{v4} = (1,1,1,1);	I_{v5} = (0,1,1);	I_{v6} = (0,1,1);
O_{v4} = ø;	O_{v5} = (1,1,2);	O_{v6} = (1,1,2);

The vertex u_1 is choosen in the graph H and the auxiliary digraph $\vec{H}(u_1)$ is constructed (see Figure A2).

The computed characteristics of vertices of the newly constructed digraph are given in the Table A2.

Table A2. The vertex characteristics of the digraph $\vec{H}(u_1)$.

I_{u1} = ø;	I_{u2} = (0,1,1);	I_{u3} = (0,1,1);
O_{u1} = (1,1,1,1);	O_{u2} = (1,1,2);	O_{u3} = (1,1,2);
I_{u4} = (0,1,1);	I_{u5} = (1,1,1,1);	I_{u6} = (0,1,1);
O_{u4} = (1,1,2);	O_{u5} = ø;	O_{u6} = (1,1,2);

It is easy to see that the constructed auxiliary directed graph $\vec{G}(v_1)$ and $\vec{H}(u_1)$ are positionally equivalent.

The vertices v_1 and u_1 are removed from the initial graphs G and H respectively. As a result, the graphs G_1 and H_1 are obtained (see Figure A3).

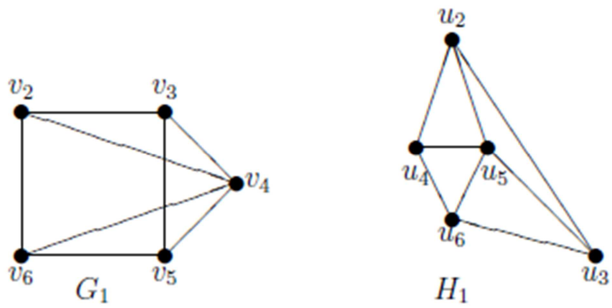

Figure A3. The graphs G_1 and H_1.

In the graph G_1, the vertex v_2 is choosen and the auxiliary digraph $\vec{G}_1(v_2)$ is constructed (see Figure A4).

The characteristics of vertices of the auxiliary digraph $\vec{G}_1(v_2)$ are computed (see the Table A3).

Table A3. The vertex characteristics of the digraph $\vec{G}_1(v_2)$.

I_{v2} = ø;	I_{v3} = (0,1);	I_{v4} = (0,1,1);
O_{v2} = (1,1,1);	O_{v3} = (1,2);	O_{v4} = (1,1,2);
I_{v5} = (1,1,1);	I_{v6} = (0,1);	
O_{v5} = ø;	O_{v6} = (1,2);	

In the graph of H_1, the vertex u_2 is choosen and the auxiliary digraph $\vec{H}_1(u_2)$ is constructed (see Figure A4).

The computed characteristics of vertices of the auxiliary digraph $\vec{H}_1(u_2)$ are given in the Table A4.

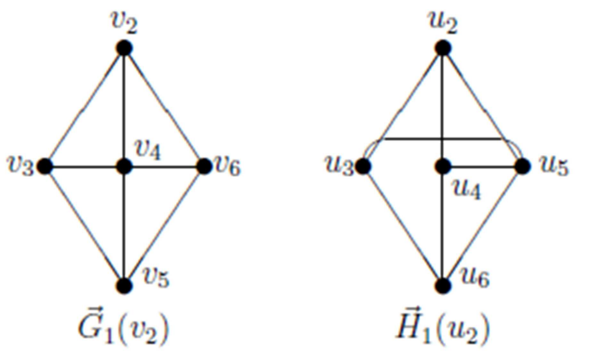

Figure A4. The auxiliary digraphs $\vec{G}(v_2)$ and $\vec{H}_1(u_2)$.

Table A4. The vertex characteristics of the digraph $\vec{H}_1(u_2)$.

I_{u2} = ø;	I_{u3} = (0,1);	I_{u4} = (0,1);
O_{u2} = (1,1,1);	O_{u3} = (1,2);	O_{u4} = (1,2);
I_{u5} = (0,1,1);	I_{u6} = (1,1,1);	
O_{u5} = (1,1,2);	O_{u6} = ø;	

Again it is found that the constructed auxiliary digraphs $\vec{G}_1(v_2)$ and $\vec{H}_1(u_2)$ are positionally equivalent.

The vertices v_2 and u_2 are removed from the graphs G_1 and H_1, respectively. As a result, the graphs G_2 and H_2 are obtained (see Figure A5).

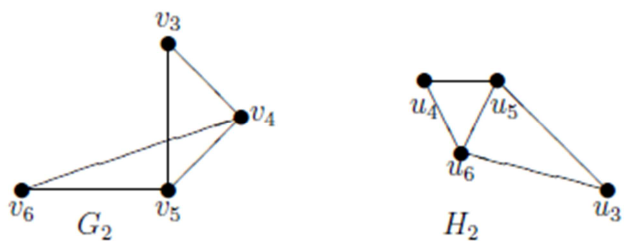

Figure A5. The graphs G_2 and H_2.

In the graph G_2, the vertex v_3 is choosen and the auxiliary digraph $\vec{G}_2(v_3)$ is constructed (see Figure A6).

The computed characteristics of vertices of the auxiliary digraph $\vec{G}_2(v_3)$ are given in the Table A5.

Table A5. The vertex characteristics of the digraph $\vec{G}_2(v_3)$.

I_{v3} = ø;	I_{v4} = (0,1);	I_{v5} = (0,1);	I_{v6} = (1,1);
O_{v3} = (1,1);	O_{v4} = (1,2);	O_{v5} = (1,2);	O_{v6} = ø;

It is constructed the auxiliary directed graph $\vec{H}_2(u_3)$ and the characteristics of its vertices are found (see the Table A6).

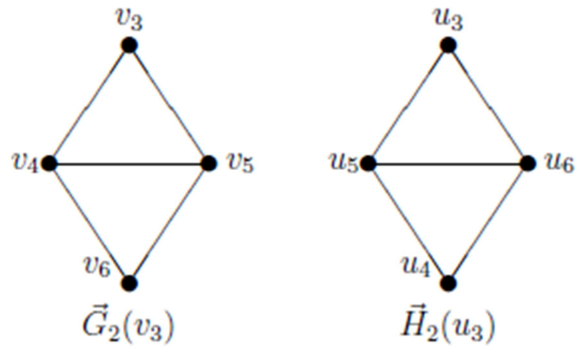

Figure A6. Auxiliary digraphs $\vec{G}_2$(v3) and $\vec{H}_2$(u3).

Table A6. The vertex characteristics of the digraph $\vec{H}_2(u_3)$.

I_{u3} = ø;	I_{u4} = (1,1);	I_{u5} = (0,1);	I_{u6} = (0,1);
O_{u3} = (1,1);	O_{u4} = ø;	O_{u5} = (1,2);	O_{u6} = (1,2);

The auxiliary digraphs $\vec{G}_2(v_3)$ and $\vec{H}_2(u_3)$ are positionally equivalent.

The vertices v_3 and u_3 are removed from the graph G_2 and H_2 respectively. As a result, the graphs G_3 and H_3 are obtained (see Figure A7).

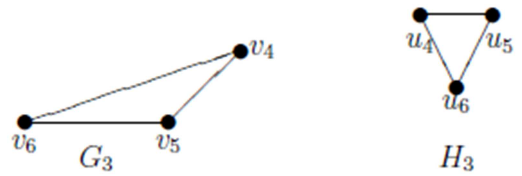

Figure A7. The graphs G_3 and H_3.

In the graph G_3, the vertex v_4 is choosen and the auxiliary digraph $\vec{G}_3(v_4)$ is constructed (see Figure A8).

Figure A8. The auxiliary digraphs $\vec{G}_3(v_4)$ and $\vec{H}_3(v_4)$.

The characteristics of the auxiliary graph $\vec{G}_3(v_4)$ are computed (see the Table A7).

Table A7. The vertex characteristics of the digraph $\vec{G}_3(v_4)$.

I_{v4} = ø;	I_{v5} = (0,1);	I_{v6} = (0,1);
O_{v4} = (1,1);	O_{v5} = (1);	O_{v6} = (1);

The auxiliary digraph $\vec{H}_3(u_4)$ is constructed and the characteristics of its vertices are found (see the Table A8).

Table A8. The vertex characteristics of the digraph $\vec{H}3(u_4)$.

I_{u4} = ø; O_{u4} = (1,1);	I_{u5} = (0,1); O_{u5} = (1);	I_{u6} = (0,1); O_{u6} = (1);

Again the constructed auxiliary digraphs $\vec{G}_3(v_4)$ and $\vec{H}_3(u_4)$ are positionally equivalent.

The vertices v_4 and u_4 are removed from the graph G_3 and H_3 respectively.

As a result, the graphs G_4 and H_4 are obtained (see Figure 10).

Figure A9. The graphs G_4 and H_4.

In the graph of G_4, vertex v_5 is choosen and the auxiliary digraph $\vec{G}_4(v_5)$ is constructed (see Figure A10).

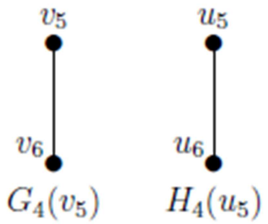

Figure A10. Auxiliary digraphs $\vec{G}_4(v_5)$ and $\vec{H}_4(u_5)$.

The characteristics of the auxiliary graph $\vec{G}_4(v_5)$ are computed (see the Table A9).

Table A9. The vertex characteristics of the digraph $\vec{G}4(v_5)$.

I_{v5} = ø; O_{v5} = (1);	I_{v6}=(0); O_{v6}= ø.

The auxiliary digraph $\vec{H}_4(u_5)$ is constructed (see Figure A10) and the characteristics of its vertices are found (see the Table A10).

Table A10. The vertex characteristics of the digraph $\vec{H}4(u_5)$.

I_{u5} = ø; O_{u5} = (1);	I_{u6}=(0); O_{u6}= ø.

Again, the auxiliary digraphs $\vec{G}_4(v_5)$ and $\vec{H}_4(u_5)$ are positionally equivalent.

The vertices v5 and u5 are removed from the graphs $G4$ and $H4$, respectively. The one-vertex graphs G_5 and H_5 are obtained. The auxiliary digraphs $\vec{G}_5(v_6)$ and $\vec{H}_5(u_6)$ contain one vertex and, of course, are the positionally equivalent.

Conclusion: the given graphs G and H are isomorphic.

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