International Journal of Bioinformatics and Biomedical Engineering
Vol. 1, No. 1, 2015, pp. 43-52
http://www.aiscience.org/journal/ijbbe

Numerical Solution of the Generalized Burgers-Huxley Equation by Exponential Time Differencing Scheme

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Abstract

Numerical solutions of nonlinear partial differential equations, such as the generalized and extended Burgers-Huxley equations which combine effects of advection, diffusion, dispersion and nonlinear transfer are considered in this paper. Such system can be divided into linear and nonlinear parts, which allow the use of two numerical approaches. Higher order finite difference schemes are employed for the spatial discretization, the resulting nonlinear system of ordinary differential equation is advanced with the modified fourth-order exponential time differencing Runge-Kutta (ETDRK4) method designed to generate the scheme with a smaller truncation error and better stability properties. The stability region of this scheme is shown and computed via its amplification factor. Numerical simulations with comparisons are presented to address any queries that may arise.

Keywords

Burgers-Huxley Equation, Exponential Time Differencing, Nonlinear, PDEs, Reaction-Diffusion, Stability

Received: May 18, 2015 / Accepted: June 6, 2015 / Published online: July 13, 2015

1. Introduction

Most of the mathematical formulations of meaningful physical system exist in the form of partial differential equations (PDEs), which arise largely in various fields such as fluid dynamics, chemical kinetics and biological models and which have since become an important and ground breaking tool for describing most real-world physical phenomena in applied areas of science, engineering and technology. As a result of their increasing importance and popularity, much attention has recently been devoted to develop an accurate and efficient method for solving systems of PDEs. The PDE that is of physical interest here is the generalized Burgers-Huxley equation which has the following form

$$u_t = Lu + N(u,t) \tag{1.1}$$

where $u = u(x, t)$ in one space dimension that represents the density or concentration of physical or biological species, $L = Du_{xx}$ and $N = -\alpha u^{\delta} u_x + \beta u(1-u^{\delta})(u^{\delta}-\gamma)$ represent the linear and nonlinear operators respectively. The linear operator L is also the term that is responsible for the stiffness and contains spatial derivatives of the highest order.

Many of time dependent PDEs (1.1) have linear terms that consist of both dispersive and dissipative terms. In addition, they are largely found combining low-order nonlinear terms with higher-order linear terms [19]. Examples of such dynamics include the Burgers, Burgers-Fisher, Cahn-Hilliard, Fishers-KPP, rotating and non-rotating Navier-Stokes, generalized Kuramoto-Sivashinsky, nonlinear Schrodinger and Burgers-Huxley equations among many others. When system (1.1) is discretized in space with central finite difference, it results to large systems of strongly nonlinear, stiff ordinary differential equations (ODEs) of the form

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$$u_t = Lu + N(u,t) \tag{1.2}$$

We can associate the time scale in each of the linear and nonlinear terms in (1.2) to the maximal rate of variation induced in the solution. This time scale depends on the solution in the case of the nonlinear term whereas, it depends on the spatial discretization of the derivatives in the PDE for the linear term. As a result, if system of ODE is solved with an explicit scheme without taking into account a small time step, it could lead to a serious numerical instability. Numerical solution of (1.2) has led many researchers to the use of various time stepping schemes. The most notable among them are the exponential time differencing (ETD) methods [6], implicit-explicit (IMEX) schemes [11, 23] and exponential integrators [12, 13, 25, 31, 32]. The IMEX exponential integrators based on multistep schemes were constructed in [2]. Cox and Matthews [6] formulated the family of exponential time differencing (ETD) schemes that were based on the Runge-Kutta and multistep methods. The ETDRK4 method with better stability properties was introduced by Krogstad [20]. To circumvent the order-reduction in case of stiff problems, Hochbruck and Ostermann [12] construct exponential integrators based on Runge-Kutta schemes. Other time stepping schemes include the split step, integrating factor and sliders methods, see [19] and references therein for details. The present paper will address the recent trends in the development of an efficient computational method that may be used to solve such and related dynamical system.

The paper is structured into 5 sections. Section 2 provides an introduction to the derivation of the fourth-order exponential time differencing Runge-Kutta (ETDRK4) method of Cox and Matthews [6] in a modified form as proposed by Kassam and Trefethen [19] with its stability analysis when applied to a system of (1.2). In Section 3, we present the generalized Burgers-Huxley equations of second and fourth order derivatives. Section 4 is based on the numerical treatments of the partial derivatives, experiments and results. Conclusions are drawn in Section 5.

2. Fourth-Order Exponential Time Differencing Runge-Kutta Method

To derive the fourth-order exponential time differencing Runge-Kutta (ETDRK4) method, we follow the notations used by Cox and Matthews [6] and describe the method in the context (1.2) by multiplying through by the term e^{-Lt}, called the integrating factor, we obtain

$$e^{-Lt}u_t = e^{-Lt}Lu + e^{-Lt}N(u,t)\,. \tag{2.1}$$

On integrating (2.1) over a single time step in the interval of length h, that is, $[t = t_n, t_{n+1} = t_n + h]$ to have

$$u(t_{n+1}) = u(t_n)e^{Lh} + e^{Lh}\int_0^h e^{-L\tau}N(u(t_n+\tau),t_n+\tau)d\tau\,. \tag{2.2}$$

Equation (2.4) is *exact* and serves as the mother scheme from which various exponential time differencing (ETD) schemes are derived, depending on how the integral is evaluated. Examples of the exponential-based methods adopting this idea of separating the linear terms were formulated in [2, 6]. A direct application of the standard fourth-order Runge-Kutta method yields a scheme known as ETDRK4 by Cox and Matthew [6], in their scheme the cancelation errors were much more pronounced which caused it to suffer serious numerical instability and order reduction in its computation.

2.1. Computational Issues

We utilize in this work the modified [19] version of the ETDRK4 scheme that has been formulated to overcome the inherent numerical stability with smaller local truncation error [20],

$$\begin{aligned}
&a_n = u_n e^{Lh/2} + (Lh/2)\phi_1(Lh/2)N(u_n,t_n),\\
&b_n = u_n e^{Lh/2} + (Lh/2)[\phi_1(Lh/2) - 2\phi_2(Lh/2)]N(u_n,t_n)\\
&+h\phi_2(Lh/2)N(a_n,t_n+h/2),\\
&c_n = u_n e^{Lh} + h[\phi_1(Lh) - 2\phi_2(Lh)]N(u_n,t_n)\\
&+2h\phi_2(Lh)N(c_n,t_n+h),
\end{aligned}$$

And

$$\begin{aligned}
u_{n+1} = u_n e^{Lh} &+ h[4\phi_3(Lh) - 3\phi_2(Lh) + \phi_1(Lh)]N(u_n,t_n)\\
&+2h[\phi_2(Lh) - 2\phi_3(Lh)]N(a_n,t_n+h/2)\\
&+2h[\phi_2(Lh) - 2\phi_3(Lh)]N(b_n,t_n+h/2)\\
&+h[\phi_3(Lh) - 2\phi_2(Lh)]N(c_n,t_n+h),
\end{aligned} \tag{2.3}$$

with define functions $\varphi_{1,2,3}$ defined as

$$\phi_1(z) = \frac{e^z - 1}{z}, \quad \phi_2 = \frac{e^z - 1 - z}{z^2}, \quad \phi_3 = \frac{e^z - 1 - z - z^{2/2}}{z^3}$$

which precisely coincide with the terms in the Lie group methods by Munthe-Kaas [28]. The functions φi have the Taylor series representation

$$\phi_j = \sum_{k=j}^{\infty} \frac{1}{k!} z^{k-j}\,. \tag{2.4}$$

The original formula of Cox and Matthews [6] contains a split-step approach which subjected it to serious numerical instability. Authors in [6] was aware of this problem and as a result, Kassam and Trefethen [19] proposed a different approach to circumvent the marginally worse error and numerical instability properties inherent in the higher-order ETDRK schemes by introducing the complex contour integrals

$$\phi_j(L) = \frac{1}{2\pi i}\int_{\Gamma} \phi_j(t)(tI - L)^{-1}\,dt \qquad (2.5)$$

to actually evaluate the coefficients in the scheme (2.3). Readers are referred to [19] for the detail derivation and implementations of the modified ETDRK4 method.

2.2. Stability of ETDRK4 Method

In this section we consider the stability of ETDRK4 method by linearizing the autonomous system of ODE (1.2) [6, 7, 15, 29], about a fixed point u_0 such that $Lu_0 + N(u(t))$, to yield

$$u_t = Lu(t) + \lambda u(t), \qquad (2.6)$$

where $u(t)$ is now the perturbation of u_0 and $\lambda = N'(u_0)$ is a diagonal or a block diagonal matrix containing the eigenvalue of N. For the fixed point u_0 to be stable, we require that $Re(L + \lambda) < 0$, for all λ. When the ETDRK4 method (2.3) is applied to (2.6), we obtain a recurrence equation

$$\frac{u_{n+1}}{u_n} = r(x,y) = L_0 + L_1 x + L_2 x^2 + L_3 x^3 + L_4 x^4, \qquad (2.7)$$

where

$$\begin{aligned}
L_1 &= 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + O(y^4),\\
L_2 &= \frac{1}{2} + \frac{y}{2} + \frac{y^2}{4} + \frac{247y^3}{2880} + O(y^4),\\
L_3 &= \frac{1}{6} + \frac{y}{6} + \frac{61y^2}{720} + \frac{y^3}{36} + O(y^4),\\
L_4 &= \frac{1}{24} + \frac{y}{32} + \frac{7y^2}{640} + \frac{19y^3}{11520} + O(y^4),
\end{aligned} \qquad (2.8)$$

which represent the respective asymptotic expansions in (2.7). It is obvious from (2.8) that as $y \to 0$, the amplification factor for the ETDRK4 reduces to

$$r(x,0) = 1 - x + \frac{x^2}{2} - \frac{x^6}{6} + \frac{x^4}{24},$$

which invariably coincides with the stability function of the standard fourth-order Runge-Kutta (RK4) methods. We observed that

$$\lim_{x,y\to 0} \partial_y r(x,y) = -1, \qquad \lim_{x,y\to 0} \partial_x r(x,y) = -1,$$

And the absolute value of the amplification factor is given as $|r(x, y)| \le 1$, which we are only able to plot. Hence, the boundary of the stability region is determined by setting $r = e^{i\theta}$, for $\theta \in [0, 2\pi]$. We plot the stability region in the complex x plane and displayed in Figure 1, where the horizontal and vertical axes represent the real and imaginary of x respectively.

Convincingly in Figure 1, we can deduce that the stability region of the ETDRK4 method grows larger as $y \to -\infty$.

3. The Models Description

Some of the numerically observed dynamics of the nonlinear PDEs contain either of dispersion or dissipation linear terms. For the dissipative PDEs, when discretized with the central finite difference schemes, the eigenvalues of the matrix operator L are negative and real, whereas they are imaginary in the case of dispersive PDEs. Dissipation in a dynamical system illustrates the important concept of mechanical modes, such as waves or oscillations, losing energy over time. Such dynamics are referred to as the dissipative systems. On the other hand, a dispersive PDE describes a system in which waves of different frequencies propagate at different phase velocities, which implies the rate at which the phase of the wave propagates in space. The present work deals with the numerical solutions in nonlinear diffusive and dispersive media.

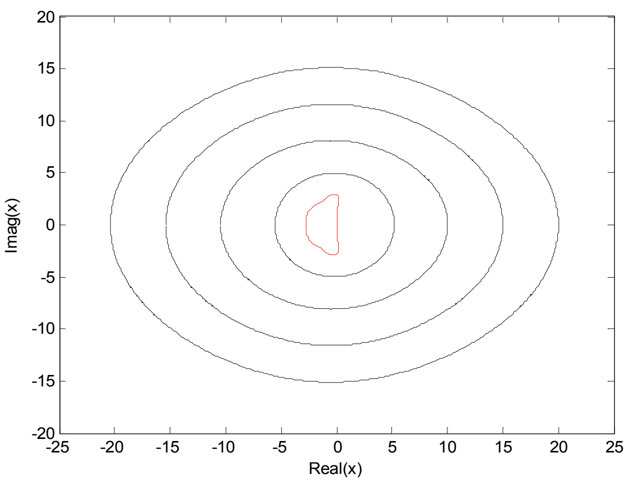


Figure 1. The boundary of stability regions of the modified ETDRK4 in the complex plane x for some negative values of y. The curves correspond to $y = -20, -15, -10, -5, 0$ from the outer curve to the inner curve. The inner curve at $y = 0$, clearly coincides with the stability region of the classical fourth-order Runge-Kutta scheme.

3.1. The Generalized Burgers-Huxley Equation

The generalized Burgers-Huxley (gBH) equation is a nonlinear PDE of second order of the form

$$\begin{aligned}&\partial_t u = Du_{xx} - \alpha u^\delta u_x + \beta u(1-u^\delta)(u^\delta - \gamma),\\ &a \le x \le b,\ t > 0,\end{aligned} \tag{3.1}$$

Where $u = u(x, t)$ is a differentiable function that depends on the spatial variable $x \in [a, b]$ and $t > 0$, satisfies the advection-diffusion system, $\alpha > 0$ describes nonlinear transfer, $\beta > 0$, $\gamma \le 1$ and $\delta > 0$ characterize a nonlinear source, with $D \ne 0$ as the diffusion coefficient. We shall restrict x to a bounded interval in this paper, in order to pave way for the numerical simulations on intervals that will not influence the behaviour of the system.

The variability of these parameters leads to many nonlinear evolution equations. For instance, when $\beta = 0, D = 1$, it reduces to the modified Burgers equation [3] that was used to describe the propagation of waves in nonlinear dissipative systems and several other physical contexts like sound waves in viscous media. Equation (3.1) reduces to the Huxley equation [14, 34] for $\alpha = 0$, $D = 1$, which is an important evolution equation with applications in biology for describing nerve propagation in nerve fibres and wall motion in liquid crystals.

When $\alpha = 1, \beta > 0$ $\delta = 2, \gamma = 0$ and $D = 1$, gives the Burgers-Huxley equation [22], which has been used to model the interaction between reaction mechanisms, convection effects and diffusion transports. Also, for $\alpha = 0$, $D > 0$ the gBH equation is the Fitzhugh-Nagumo equation [9, 14], and when $\alpha = 0$, $\delta = 1$, $\gamma = -1$ and $D = 1$ it is known as the Newell-Whitehead equation [24].

Numerical solutions of the nonlinear differential equations became paramount since there is no specific method of finding the analytical solutions or in some situations where the analytical results of the nonlinear diffusion equations are too involved to be useful. A number of researchers have used both analytical and numerical techniques to solve the gBH equation. Some of the analytical studies among others include, the homotopy analysis methods of Wazwaz [35] and Molabahrami [26] to solve Burger-huxley equation, Deng [8] employed the first integral method to study the travelling wave solutions for the gBH equation, and the solitary wave solutions by Wang et al. [33]. On the other hand, Soheili et al. [30] applied a moving mesh PDE technique for Burgers-type nonlinear equations. A fourth order improved numerical scheme was used to solve gBH equation by Bratsos [4], other numerical methods include the Adomian decomposition method [16], the pseudospectral and spectral collocation methods [17, 18] and the variation iteration technique [1] to solve the gBH equation.

By using the factorization method [5], we apply the travelling wave transformation $u(x,t) = v(\xi)$, with $\xi = k(x - ct)$, where k, c are the respective wavenumber and wave velocity. Then, the gBH is converted to an ODE form.

$$\frac{d^2v}{d\xi^2} + \frac{(c - \alpha v^\delta)}{k}\frac{dv}{d\xi} + \frac{\beta(1-v^\delta)(v^\delta - \gamma)}{k^2} = 0. \tag{3.2}$$

For simplicity, we let $\omega(v) = (1-v^\delta)(v^\delta - \gamma)\beta k^{-2}$ so that

$$\begin{aligned}&\frac{d\omega_1}{dv}v + \omega_1 + \omega_2 = \frac{\sqrt{\beta}}{k}\left[-\tau\delta v^\delta + \tau(1-v^\delta) + \frac{(v^\delta - \gamma)}{\tau}\right]\\ &+\frac{1}{k}(c - \alpha u^\delta) = 0\end{aligned} \tag{3.3}$$

where $\omega_1(v) = \tau\sqrt{\beta}/k(1-v^\delta)$, $\omega_2(v) = \tau^{-1}\sqrt{\beta}/k(v^\delta - \gamma)$ and τ, a constant. On simplifying we get

$$\sqrt{\beta}\left[(-\tau + \tau^{-1})v^\delta - \alpha v^\delta + (\tau - \frac{\gamma}{\tau})\right] + c = 0, \tag{3.4}$$

where

$$c = -\sqrt{\beta}\left(\tau - \frac{\gamma}{\tau}\right), \qquad \tau = \alpha \pm \sqrt{\alpha^2 + 4\beta(1+\delta)}(2\sqrt{\beta(1+\delta)})^{-1}.$$

By adopting the grouping technique used in [5, 10], we have that

$$\begin{aligned}&\frac{d^2v}{d\xi^2} + \frac{1}{k}\sqrt{\beta}\left[\left(\tau - \frac{\gamma}{\tau}\right) - (\tau(1+\delta) - \frac{1}{\tau})v^\delta\right]\frac{dv}{d\xi}\\ &+\omega_1\omega_2(v) = 0\end{aligned} \tag{3.5}$$

with an equivalence first-order system

$$\frac{dv}{d\xi} \pm \frac{\tau\sqrt{\beta}(1-v^\delta)}{k}v = 0, \tag{3.6}$$

which on integration results to a general solution

$$v(x,t) = \left\{1 \pm C\exp\left[-\left(\frac{-\alpha \pm \sqrt{\alpha^2 + 4\beta(1+\delta)}}{2(1+\delta)k}\right)\delta\xi\right]\right\}^{-1/\delta} \tag{3.7}$$

with C, an arbitrary constant, we shall use this result later.

Again, if we substitute $\psi = u^\delta$, $k = 1$ and $D = 1$ into (3.1), we have

$$\begin{aligned}&(\alpha\psi^2 - c\psi)\psi' + (\delta - 1)\frac{\delta'^2}{\delta} - \psi\psi''\\ &-\beta\delta\psi^2(1-\psi)(\psi - \gamma) = 0\end{aligned} \tag{3.8}$$

Hence, Eq. (3.8) can further be written in the form coupled nonlinear autonomous ODEs

$$\psi' = \psi X,$$
$$X' = (\alpha\psi - c)X - X^2\delta^{-1} - \beta\delta(1-\psi)(\psi-\gamma) \quad (3.9)$$

where c is the wave velocity, X, ψ' and ψ'' denote $\frac{\psi'}{\psi}, \frac{d\psi}{d\xi}$ and $\frac{d^2\psi}{d\xi^2}$ respectively.

3.2. The Extended Burgers-Huxley Equation

Consider the extended Burgers-Huxley (eBH) equation [22]

$$\partial_t u = -\sigma 2 u_{xxxx} + u_{xx} - \alpha u^{\delta} u_x + \beta u(1-u^{\delta})(u^{\delta}-\gamma),$$
$$a \le x \le b,\ t > 0, \quad (3.10)$$

subject to the initial condition $u(x, 0) = u0(x)$, and the periodic boundary conditions at $x = a, b$. The parameters $\alpha, \beta, \delta, \sigma$ are real and positive, with $\gamma \le 0$. The eBH equation has application in population dynamics [27] as a result of the long-range effect arising from the fourth-order derivative term. When $\sigma = 0$, it reduces to the gBH equation.

We seek for the traveling wave solution of the form $u(x, t) = U(\xi)$, where $\xi = x - ct$, and put it in (3.10) to yield

$$-\sigma^2 U'''' + U'' - \alpha U U' + cU' - \beta U(U-1)(U-\gamma) = 0. \quad (3.11)$$

For simplicity, we let $\delta = 1$, $U' = \upsilon$, $\upsilon' = \mu$ and $\sigma\mu' = \rho$, we reduce (3.11) to an equivalent coupled ODEs system

$$U' = \upsilon,\ \upsilon' = \mu,\ \sigma\mu' = \rho,$$

$$\sigma\rho' = \mu - \alpha U\upsilon + c\upsilon - \beta U(U-1)(U-\gamma). \quad (3.12)$$

It is known that the gBH equation (3.1) takes the form of travelling wave solutions (3.7) connecting the two equilibrium points, say $\hat{u}_0 = (U, v) = (0,0)$ and $\hat{u}_1 = (U, v) = (1,0)$, at $\delta = 1$. This actually guides on the restriction of $x \in [a, b]$, with the boundary conditions

$$X(0, t) = a,\ x(1, t) = b. \quad (3.13)$$

Similarly, for the eBH equation, the equilibrium steady states for system (3.12) are $\hat{u}_0 = (u, \upsilon, \mu, \rho) = (0, 0, 0, 0)$ and $\hat{u}_1 = (u, \upsilon, \mu, \rho) = (1, 0, 0, 0)$. The linearization at the neighbourhood of the steady state $\hat{u} = 0$ results to the characteristic equation

$$\sigma^2\lambda^4 - \lambda^2 - c\lambda + \beta\gamma = 0 \quad (3.14)$$

which arise from the Jacobian or community matrix of system (3.12). Likewise, we linearize around the steady state $\hat{u} = 1$and obtain

$$\sigma^2\lambda^4 - \lambda^2 + (\alpha - c)\lambda + (1-\gamma)\beta = 0. \quad (3.15)$$

In each of the cases, if $\sigma = 0$, the characteristics equation is reduced to that of the gBH equation with two roots in the complex half-plane. For $\sigma > 0$, but sufficiently small, the connection between the two critical points is heteroclinic.

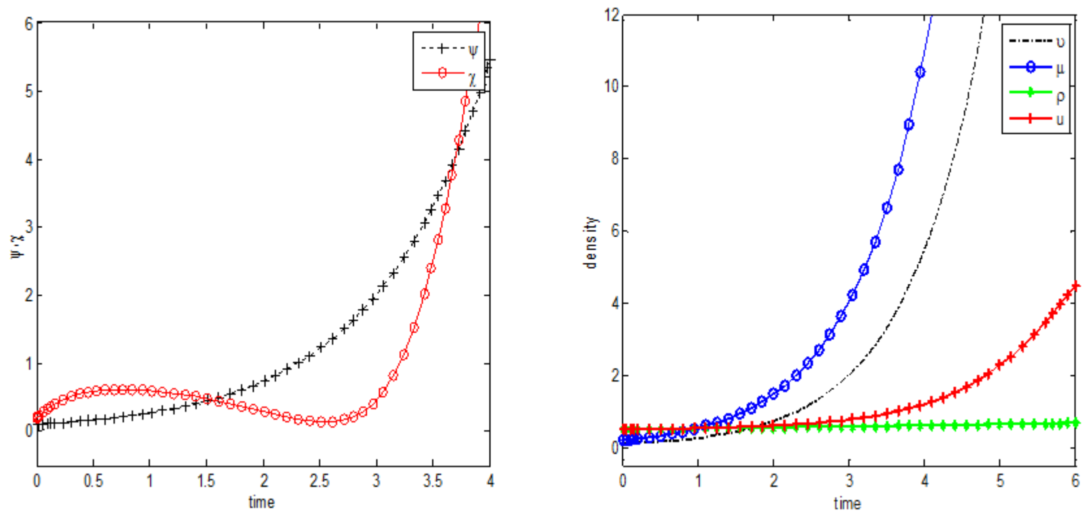


Figure 2. Behaviours of the systems (3.9) and (3.12) near their stationary states obtained at parameter values: $\alpha = 0.5$, $\beta = 2$, $\gamma = 1$, $\delta = 1.2$, $c = 2$ and $t = 4$ for the left plot; $\alpha = 0.01$, $\beta = 0.5$, $\sigma = 0.05$, $c = 2\sqrt{-\beta\gamma}$ and $\gamma = -1$ for the right plot.

4. Numerical Treatments and Experiments

Our goal in this paper is to solve the time dependent PDEs discussed in Section 3 by means of finite difference scheme and the ETDRK4 method whose derivation and stability have been presented in Section 2. We shall employ the method of lines (MOL) technique in which the spatial derivatives in these equations are discretized with the fourth-order central finite difference operators to convert such system to an ODE of the form (1.2).

Also in this section, we let parameters M and N be positive integers, a, b be real numbers in such that $a < b$, and $T > 0$. To approximate the solutions of the gBH and eBH equations in the spatial interval $x \in [a, b]$ over time T, we discretize the spatial domain by a uniform partition $a = x_0(t) < x_1(t) < x_2(t) < \cdots < x_{N-1}(t) < x_N = b$, where a, b are the two end points, and $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$ for the initial and final time in $[0, T]$, we let the step size $\Delta x = h = (b - a)/N$ and time step $\Delta t = k = T/M$.

The fourth-order central finite difference approximations used for the first, second and fourth order partial derivatives are:

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_i} \approx \frac{u_{1-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}}{12h} = A, \tag{4.1}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_i} \approx \frac{-u_{1+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12h^2} = B, \tag{4.2}$$

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_{x=x_i} \approx \frac{u_{i-3} + 12u_{i-2} - 39u_{i-1} + 56u_i - 39u_{i+1} + 12u_{i+2} - u_{i+3}}{6h^4} = C, \tag{4.3}$$

where $u(x_i, t) \approx u_i(t)$, $0 \le i \le N$, for i indicating a position along the in x, A, B and C, are the corresponding Toeplitz matrices of size $(N - 1) \times (N - 1)$.

Using the above approximations, the semi-discretized form of (3.1) in terms of the matrices A and B becomes

$$\frac{du}{dt} = DBu - \alpha u^{\delta} Au + \beta u(1 - u^{\delta})(u^{\delta} - \gamma). \tag{4.4}$$

Similarly, the semi-discretized form of (3.10) in terms of these three matrices becomes

$$\frac{du}{dt} = -\sigma^2 Cu + Bu - \alpha u^{\delta} Au + \beta u(1 - u^{\delta})(u^{\delta} - \gamma), \tag{4.5}$$

where $u = [u_1, u_2, \ldots, u_{N-1}]^T$ the parameters α, β, δ, σ and D are real and positive and $\gamma \le 1$. Equations (4.4) and (4.5) are now systems of nonlinear ODEs, they require to be advanced with the time stepping (ETDRK4) method as discussed in Section 2.

4.1. Problem 1

Case I: For the first example, let us consider a specific case of the gBH Eq. (3.1) when $\alpha = 1$, $\beta = 2/3$, $\delta = 2$, $\gamma = 0$ and $D = 1$, which has a spectrum of applications in nonlinear physics and physiology [10, 26, 33], subject to the boundary conditions.

$$u(0, t) = a, \quad u(1, t) = b.$$

On setting $C = 1$ in (3.7) and applying the inverse transformation $v(\xi) = u(x, t)$, where

$\xi = k(x - ct)$, we obtain the exact solution

$$\frac{du}{dt} = -\sigma^2 Cu + Bu - \alpha u^{\delta} Au + \beta u(1 - u^{\delta})(u^{\delta} - \gamma), \tag{4.6}$$

and the corresponding initial condition

$$u(x,0) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{3}x\right)\right]^{\frac{1}{2}}. \tag{4.7}$$

Thus, the error function is defined by

$$error = \left|\overline{u_i} - u_i\right|, \quad 0 \le i \le N, \tag{4.8}$$

Where $\overline{u_i}$ and u_i are the exact and approximate solutions respectively. From the experiments, we present the numerical results in both 2D and 3D as seen in Figure 3, using the boundary values $x \in [a, b] = -15 \le x \le 10$ for all time levels used. This boundary condition gives more accurate results because it provides more space for the traveling waves to propagate. Also, the proposed method has given more accurate solutions at all-time levels when compared with the standard Matlab ode15s solver; this is evident in Table 1.

Table 1. Problem 1, case I. Comparisons of the proposed method for various values of x and t with $\alpha = 1$, $\beta = 2/3$, $\delta = 2$, $\gamma = 0$, $D = 1$ and $h = 0.1256$.

Method	x	t=0.01	t=0.05	t=0.10	t=1.00
	0.1256	8.4927E-06	4.2558E-05	8.5353E-05	8.9774E-04
ETDRK4	0.5024	1.0041E-05	5.0317E-05	1.0091E-04	1.0614E-03
	1.0048	1.1871E-05	5.9489E-05	1.1931E-04	1.2549E-03
	0.1256	9.4856E-04	1.1308E-03	1.3490E-03	4.6702E-03
ode15s	0.5024	2.3933E-03	2.7632E-03	3.1778E-03	7.3718E-03
	1.0048	4.0915E-03	4.6333E-03	5.2042E-03	1.0150E-02

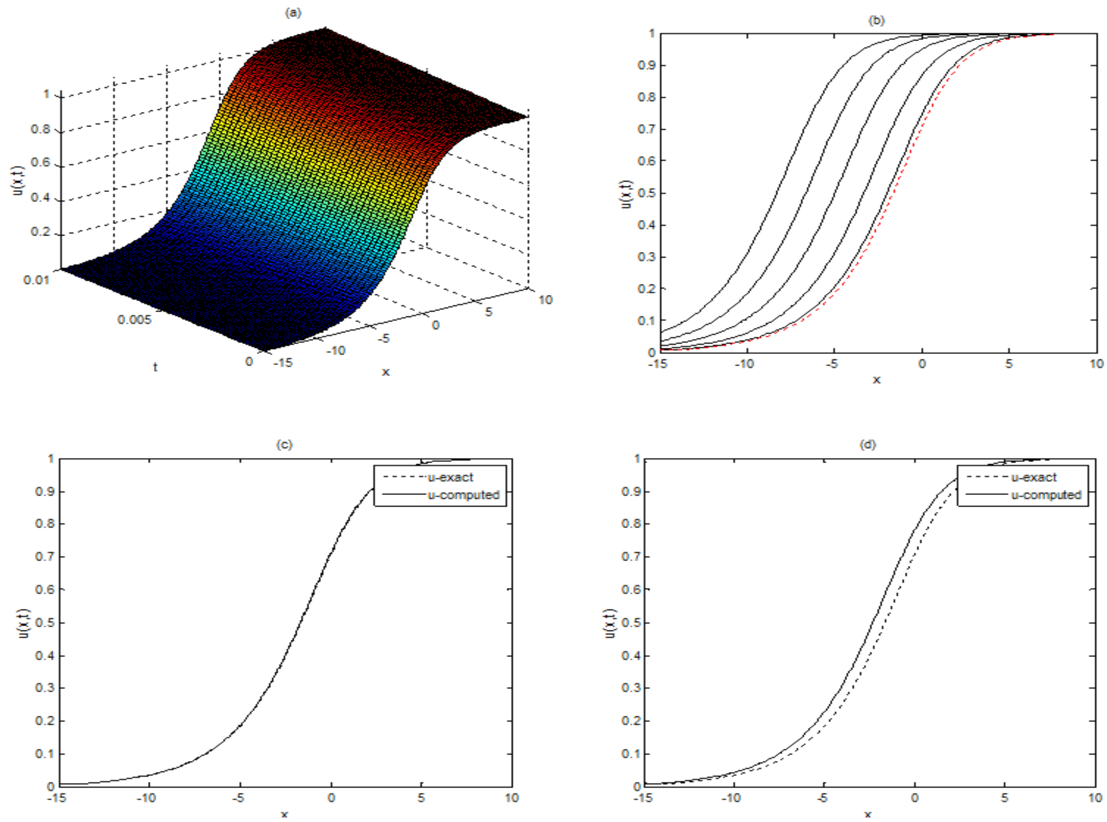


Figure 3. Numerical solution of the generalized Burgers-Huxley equation (3.1) with parameters $a = -15$, $b = 10$ for $N = 200$. (a) Showing the 3D plot at $D = 0.5$, $t = 0.01$; (b) Presents 2D plot for $t = 1, 5, 10, 15, 20$ from right to left. The solid and dotted lines in (c) and (d) denote the numerical and exact solutions at $t = 0.1$ and $t = 2$ respectively.

Case II: For the second case, we consider the gBH Eq.(3.1). The exact solution of this equation subject to the initial condition [30],

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(a_1 x\right)\right]^{\frac{1}{\delta}}, \qquad (4.9)$$

And the boundary conditions

$$\begin{aligned} u(0,t) &= \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(-a_1 a_2 t\right)\right]^{\frac{1}{\delta}}, \\ u(1,t) &= \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(a_1\left(1 - a_2 t\right)\right)\right]^{\frac{1}{\delta}}, \end{aligned} \qquad (4.10)$$

Where

$$a_1 = \frac{-\alpha\delta + \delta\sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)}\gamma,$$

$$\alpha_2 = \frac{\alpha\gamma}{1+\delta} - \frac{(1+\delta-\alpha)(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)})}{2(1+\delta)},$$

Is (see [16])

$$u(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(a_1\left(x - a_2 t\right)\right)\right]^{\frac{1}{\delta}}. \qquad (4.11)$$

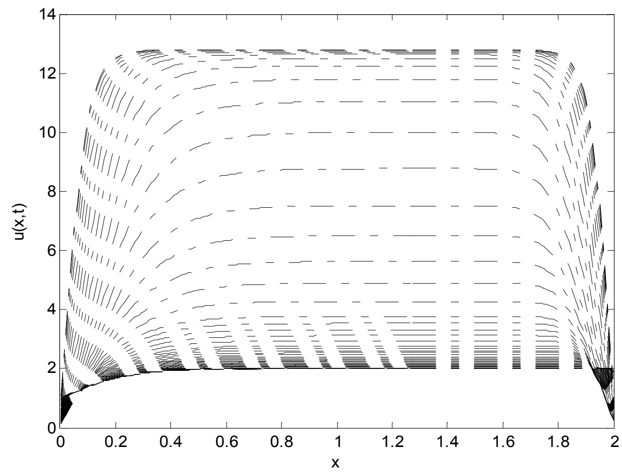


Figure 4. Contour plot showing the numerical solution of the generalized Burgers-Huxley equation (3.1) for the case II at parameter values $\alpha = 3$, $\beta = 5$, $D = 1$, $\delta = 1$ and $\gamma = 2$ at time level $t = 0.02$ for $N = 200$.

In Figure 4, we numerically observed that different boundary layers emerge as a result of the boundary conditions in (4.10). For instance, when $D = 0.5$, $\alpha = 0.03$, there is a rise in amplitude of u which results to a sharp gradient with flat boundary layer. With increments D = 1 and α = 3, the flat roof is disappearing into a curve with gentle or mildly steep slope. Readers are to take note of the variations in their amplitudes.

Case III : We apply the present method for the third case of Eq. (3.1), subject to the initial condition

$$u(x,\ 0) = \sin(\pi x), \tag{4.12}$$

and the zero-flux boundary conditions

$$u(0,\ t) = u(1,\ t) = 0. \tag{4.13}$$

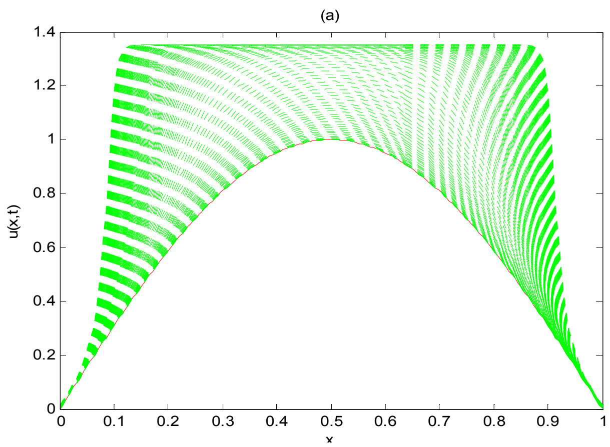
The exact solution of (3.1) for D = 1 is given by (see [8, 21])

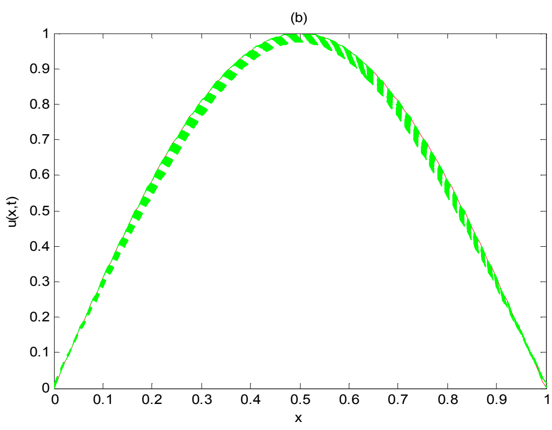
$$u(x,t) = \left\{ \frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left[\sigma\gamma\left(x - \left(\frac{\gamma\alpha}{1+\delta} - \frac{(1+\delta-\gamma)(\rho-\alpha)}{2(1+\delta)}\right)t\right)\right]\right\}^{\frac{1}{\delta}} \tag{4.14}$$

Where

$$\sigma = \frac{\delta(\rho-\alpha)}{4(1+\delta)},\ \rho = \sqrt{\alpha^2 + 4\beta(1+\delta)}\ .$$

This particular case is solved for different values of α, β, δ and D.





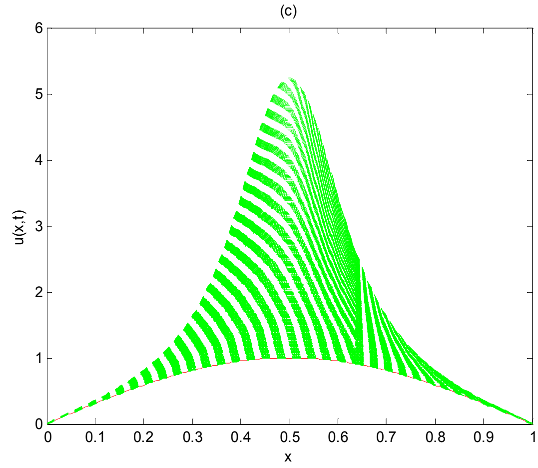


Figure 5. Numerical solution of the generalized Burgers-Huxley equation (3.1) for the *case III* for (a) α = 0.5, β = 60, D = 0.1, δ = 2, γ = 0.001, t = 0.005; (b) α = 0.1, β = 1, D = 1, δ = 2, γ = 0.001, t = 0.01; (c) α = 1, β = 1, D = 1, δ = 2, γ = 0.001, t = 0.01 for N = 200 and x in [0, 1]. Notice the scales on the vertical axes for different amplitudes in the panels.

As seen in Figure 5, by redefining the parameter values, the flat boundary layer in panel (a) where the effect of diffusion appears to have dominance over the effect of reaction, the top flat boundary layer with sharp gradient goes down to a curve with gentle slope in panel (b). It worth mentioning that at certain parameter values, solution patterns may mimic each other. For instance, at α = 1, β = 1, D = 1, δ = 2, γ = 0.001, t = 0.01 for N = 200, the solution for the case III has a close-like shape with that of the case II of Figure 4 where the varying effects of α is verified.

4.2. Problem 2

Table 2. Problem 2, Relative errors of the ETDRK4 method with respect to a gold-standard run reported at t = 1, 2 for β = 1, β = 2 and β = 3 at various time-step k. Other parameter values include α = 0.5, δ = 2, γ = −1, σ = 0.01, N = 200 and x in [−50, 50].

time	*k*	β = 1	β = 2	β = 3
t=1	1/8	3.6577E-05	4.6690E-04	2.7611E-03
	1/16	2.3133E-06	2.1550E-05	1.1148E-04
	1/32	2.1376E-07	1.1582E-06	5.7105E-06
	1/64	1.4732E-08	6.7863E-08	3.2225E-07
	1/128	9.3333E-10	4.1050E-09	1.9131E-08
	1/256	5.8012E-11	2.5251E-10	1.1654E-09
	1/512	3.6560E-12	1.5482E-11	7.1659E-11
	1/1024	3.4316E-13	1.0599E-12	4.4951E-12
t=2	1/8	3.3539E-05	4.5309E-04	2.8074E-03
	1/16	1.7944E-06	2.1919E-05	1.1446E-04
	1/32	1.0339E-07	1.1835E-06	5.8207E-06
	1/64	6.2005E-09	6.8980E-08	3.2932E-07
	1/128	3.7976E-10	4.1629E-09	1.9798E-08
	1/256	2.3616E-11	2.5586E-10	1.2138E-09
	1/512	1.3552E-12	1.5648E-11	7.4582E-11
	1/1024	5.2300E-13	1.2077E-12	4.8528E-12

For the second problem, we experiment numerically the eBH (3.10), subject to initial and boundary conditions as in case I. The spatial interval [a, b] is adjusted in order to ensure that the waves have sufficient space to propagate. The efficiency and applicability of the ETDRK4 scheme when applied to (3.10) is further justified by reporting the relative errors defined by

$$\textit{relative error} = \frac{\max|\bar{u}_i - u_i|}{\max|\bar{u}_i|}, \tag{4.15}$$

Where $\bar{u}_i$ and ui are the exact and numerical values of u at point i in the collocation interval of points

$$\{x_1=a,\ldots,x_i=a+(i-1)h,\ldots,x_N=b\}, \text{ for } h=\frac{|b-a|}{N-1}. \tag{4.16}$$

The results presented in Table 2 demonstrate the performance of the computational method with respect to a gold-standard run obtained with ETDRK4 scheme, using time step $\Delta t = k = 1/2500$ as the exact solution and compared with various k. The value of β is varied for when $t = 1$ and $t = 2$, and in both cases, it is obvious that the proposed method performs better for $\beta = 1$. The surface and contour plots for problem 2 are shown in Figure 6.

5. Conclusions

In this paper, we have studied the numerically observed dynamics of the generalized and extended Burgers-Huxley equations in one-dimensional space. The resulting nonlinear ODEs arising from their spatial discretization with fourth order central finite difference schemes were utilized by the modified fourth-order exponential time differencing Runge-Kutta scheme. In comparison, the method proposed in this paper for the family Burgers-Huxley problems is computationally efficient and reliable with an improvement over the ode15s solver. It is worth mentioning that the method presented here can easily be extended to solve n−dimensional models involving nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection terms.

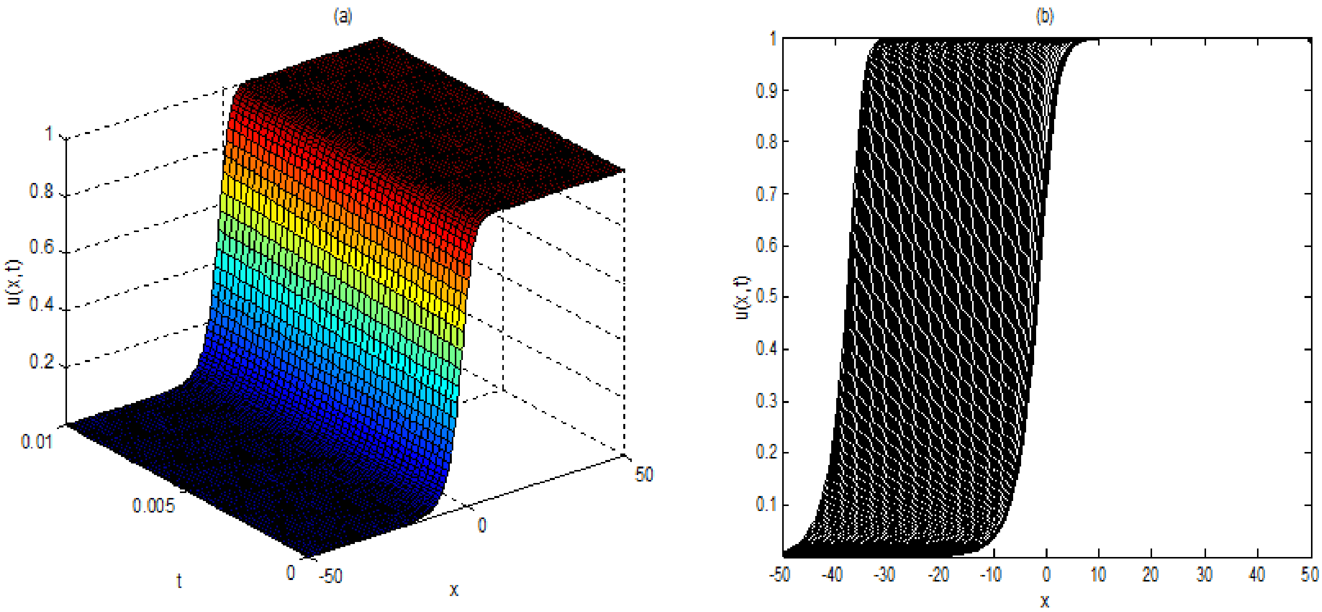


Figure 6. Numerical solution of the extended generalized Burgers-Huxley equation (3.10) (a) $\alpha = 0.5$, $\beta = 60$, $D = 0.01$, $\delta = 2$, $\gamma = 0.001$, t=0.05 and (b) $\alpha = 1$, $\beta = 1$, $D = 1$, $\delta = 1$, $\gamma = -1$, t=0.005.

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