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Bi-Section Algorithm for Solving Linear Bi-Level Programming Problem

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Abstract

The bi-level programming problem (BLPP) is significant because of its application in several areas such as transportation, finance, management, computer science and so on. This problem is an appropriate tool to model these real problems. It has been proven that the general BLPP is an NP-hard problem. Therefore, the BLPP is a practical and complicated problem so solving this problem would be significant. In this paper, we attempt for developing an algorithm based on bi-section algorithm to solve BLPP. Using the Karush-Kuhn-Tucker conditions the bi-level programming problem is converted to a non-smooth single level problem, and then it is smoothed by heuristic method for using proposed bi-section algorithm. The smoothed problem is solved using bi-section algorithm which is an exact method for solving the problem. The presented approach achieves an efficient and feasible solution in an appropriate time which has been evaluated by solving test problems.

Keywords

Bi-Section Algorithm, Linear Bi-Level Programming Problem, Karush-Kuhn-Tucker Conditions

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1. Introduction

It has been proven that the bi-level programming problem (BLPP) is an NP-Hard problem [1, 2]. Several algorithms have been proposed to solve BLPP [11, 12, 13, 21, 25, 26, 27, 28, 29, 31, 32, 33]. These algorithms are divided into the following classes: global techniques, enumeration methods, transformation methods [3, 4, 22, 23], meta heuristic approaches, fuzzy methods [5, 6, 7, 8, 24], primal-dual interior methods [13]. In the following, these techniques are shortly introduced.

1.1. Global Techniques

All optimization methods can be divided into two distinctive classes: local and global algorithms. Local ones depend on initial point and characteristics such as continuity and differentiability of the objective function. These algorithms search only a local solution, a point at which the objective function is smaller than at all other feasible points in vicinity. They do not always find the best minima, that is, the global solution. On the other hand, global methods can achieve global optimal solution. These methods are independent of initial point as well as continuity and differentiability of the objective function [9, 10, 11, 12, 34, 35].

1.2. Enumeration Methods

Branch and bound is an optimization algorithm that uses the basic enumeration. But in these methods we employ clever techniques for calculating upper bounds and lower bounds on the objective function by reducing the number of search steps. In these methods, the main idea is that the vertex points of achievable domain for BLPP are basic feasible solutions of

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the problem and the optimal solution is among them [14].

1.3. Meta Heuristic Approaches

Meta heuristic approaches are proposed by many researchers to solve complex combinatorial optimization. Whereas these methods are too fast and known as suitable techniques for solving optimization problems, however, they can only propose a solution near to optimal. These approaches are generally appropriate to search global optimal solutions in very large space whenever convex or non-convex feasible domain is allowed. In these approaches, BLPP is transformed to a single level problem by using transformation methods and then meta- heuristic methods are utilized to find out the optimal solution [15, 16, 17, 18, 19, 25, 36-40].

However there are several approaches to solve optimization problems, but there is no any exact classic approach. In this paper, the authors have tried to proposed bi-section algorithm, which is a convergent approach, to solve linear BLPP.

The remainder of the paper is structured as follows: problem formulation and smoothing method to the BLPP are introduced in Section 2. The algorithm based on bi-section is proposed in Section 3. Computational results are presented for our approaches in Section 4. As result, the paper is finished in Section 5 by presenting the concluding remarks.

2. Problem Formulation and Smoothing Method

It has been proven that the bi-level programming problem (BLPP) is an NP-Hard problem [1, 2]. Several algorithms have been proposed to solve BLPP [11, 12, 13, 21, 25, 26, 27, 28, 29, 31, 32, 33]. These algorithms are divided into the following classes: global techniques [9, 10, 11, 12, 34, 35], enumeration methods [14], transformation methods, meta heuristic approaches, fuzzy methods [5, 6, 7, 8, 24], primal-dual interior methods [13]. In the following, these techniques are shortly introduced.

The BLPP is used frequently by problems with decentralized planning structure. It is defined as [20]:

$$\begin{aligned} &\min_x F(x,y) = a^T x + b^T y \\ &\quad \text{s.t} \\ &\min_y F(x,y) = c^T x + d^T y \\ &\quad \text{s.t} \\ &\qquad Ax + By \le r, \\ &\qquad x, y \ge 0. \end{aligned} \tag{1}$$

Definition 2.1:

Every point such as (x^*, y^*) is an optimal solution to the bi-level problem if

$$F(x^*. y^*) \le F(x, y) \; \forall \, (x, y) \in IR. \tag{2}$$

Which IR is feasible region of the BLPP.

Using KKT conditions problem (1) can be converted into the following problem:

$$\begin{aligned} &\min_x F(x,y) = a^T x + b^T y \\ &\quad \text{s.t} \quad \mu B = -d, \\ &\qquad\quad \mu(Ax + By - r) = 0, \\ &\qquad\quad Ax + By - r \le 0, \\ &\qquad\quad x, y, \mu \ge 0. \end{aligned} \tag{3}$$

To convert the inequality constraint to an equality constraint, the positive slack variable v is added:

$$\begin{aligned} &\min_x F(x,y) = a^T x + b^T y \\ &\quad \text{s.t} \quad \mu B + d = 0, \\ &\qquad\quad \mu(Ax + By - r) = 0, \\ &\qquad\quad Ax + By - r + v = 0, \\ &\qquad\quad x, y, \mu, v \ge 0. \end{aligned} \tag{4}$$

3. Bi-section Algorithm (BA)

Most of nonlinear equations are very difficult to solve and some of them are unsolved. Therefore several methods have been proposed to approximate of the root these nonlinear equations. One of the most important methods to approximate the root is the bi-section algorithm. Necessary condition to use these methods, to find approximate of root, is the uniqueness root of the nonlinear equation [41, 42]. Firstly we proposed necessary definitions and theorems then explain the bi-section algorithm. Finally to illustrate this method we solve several examples.

Definition 3.1:

$\overline{A} = A \cup A'$, that A' is limit points of A.

Definition 2:

A, B are separated if $\overline{A} \cap B = \varphi, \overline{B} \cap A = \varphi.$

Definition 2.2:

A is a connected set if it is not union of two separated sets.

The following theorems guarantee that the equation $f(x) = 0$

has just a root.

Theorem1 [43]

If f be a continuous function in closed interval [a,b] and f(a)f(b)<0 then f(x)=0 has at least a root in (a,b).

Proof:

Because f is continuous then f([a,b]) is connected [3]. Let $E = f([a,b])$ if there is no any x so that f(x)=0 then $0 \notin f([a,b])$.

Now let $A = E \cap (-\infty,0)$, $B = E \cap (0,+\infty)$, $f(a) \in A$, $f(b) \in B$, $E = A \cup B$.

Also A , B are separated then E is not connected and this is opposite assumption. Then proof is complete.

Now we proposed a theorem without proof.

Theorem 2 [43]

If f be a continuous function in closed interval [a,b] and differentiable function in (a,b) then f(x)=0 has at most a root in (a,b).

Example 1:

Let $f(x) = \sin x + x \; in \; [-\frac{\pi}{4}, \frac{\pi}{4}]$.

We check two theorems to this function.

f is continuous function and $f(-\frac{\pi}{4})f(\frac{\pi}{4}) < 0$ because:

$$f(-\frac{\pi}{4}) = \sin(-\frac{\pi}{4}) - \frac{\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\pi}{4} < 0,$$

$$f(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) + \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\pi}{4} > 0.$$

Then $f(x) = \sin x + x$ has at least one root in given interval according theorem 1.

$f'(x) = \cos x + 1$ Obviously $f'(x) > 0 \, in [-\frac{\pi}{4}, \frac{\pi}{4}]$ therefore f has at most one root in given interval.

The bisection method is a simple method to find a root of equation in an interval. Suppose f is a continuous function in interval [a,b] and f(a)f(b)<0, we want approximate root of f(x) = 0. The root of f(x) =0 in interval [a,b] should be unique namely two theorem 1 and 2 should applicable to the f(x)=0 in interval [a,b]. In this method, first of all the interval [a,b] is bisected, let $c = \frac{a+b}{2}$, at the next step we check signs of f(a)f(c) and f(c)f(b), if f(a)f(c)<0 then the subinterval [a,c] is selected and we sets b=c and if f(c)f(b)<0 the subinterval [c,b] is selected and we sets a=c. Definitely in both of two cases, the sign of f(a)f(b) is negative therefore the bisection algorithm is applicable to new interval [a,b]. This procedure is continued while one of the under conditions is true.

1. $|p_{n+1} - p| < \varepsilon$, which p_{n+1}, p are the midpoint of the interval in (n+1)th step and root respectively.

2. $|p_{n+1} - p_n| < \varepsilon$, which p_{n+1}, p_n are the midpoint of the interval in (n+1)th and nth step.

3. $|f(p_n)| < \varepsilon$, which ε is a given very small and positive number in all conditions.

However bisection algorithm is a simple method, but it obtain solution very slowly. Therefore an approximation of solution is obtained by this method then it is used in other methods as a starting point.

In the bi-section method a sequence is composed so that limit of the sequence is equal to root of the given equation.

Steps

Step1: input a,b.

Step 2: let x = (a+b)/2 : print x.

Step 3: if ABS (f(x)) <EPS then end.

Step 4: if f (a) f(x)<0 then let b=x ELSE let a=x.

Step 5: GOTO 2.

Step 6: END.

Theorem 3

The bi-section method is convergent in the interval [a,b] if f(a)f(b)<0 and f is continuous.

Proof: if p_i is the midpoint of the interval in the ith step and p is solution, then absolute error in iterations is:

$$|p_1 - p| < \frac{b-a}{2}$$

$$|p_2 - p| < \frac{\frac{b-a}{2}}{2} = \frac{b-a}{2^2}$$

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$$0 \leq |p_n - p| < \frac{b-a}{2^n}$$

As we now $\lim_{n \to \infty} \frac{1}{2^n} = 0$, Consequently we have:

$$\lim_{n \to \infty} \frac{1}{2^n} = 0 \Rightarrow \lim_{n \to \infty} \frac{b-a}{2^n} = 0 \Rightarrow \lim_{n \to \infty} |p_n - p| = 0 \Rightarrow \lim_{n \to \infty} p_n = p.$$

Therefore the composed sequence by the bi-section algorithm is convergent to the root f.

Example 2

Suppose we are going to solve the simple equation by the bi-section algorithm.

$$x^2 + x = 1$$

To solve this equation firstly we manipulate it that right side be zero. Then we have

$$x^2 + x - 1 = 0$$

Equivalently the goal is finding root of function

$$f(x) = x^2 + x - 1.$$

Now we guess two number such as a,b so that $f(a)f(b) < 0$. Let a=0 and b=1 then f(a)=-1 and f(b)=1 therefore $f(a)f(b) = (-1)\times 1 < 0$.

Because f is polynomial then it is continuous in every interval of real numbers particularly in [0, 1]. Therefore f has conditions of theorem 1 then f has at least one root in [0, 1].

Derivative of f equal to:

$$f'(x) = 2x + 1$$

Obviously $f'(x)$ is positive in (0, 1) therefore $f'(x) > 0$ then f has conditions of theorem 2 too. Namely f(x) =0 has at most a root in (0, 1).

According to the theorem 1 and theorem 2, f(x) =0 has just one root in (0, 1). Now we con use the bi-section algorithm to find the root of $f(x) = x^2 + x - 1$ in interval [0, 1].

The following table is summary of the bi-section method to solve this example at the five iterations:

Table 1. Stages of bi-section algorithm-Example 2.

Iterations	a	b	x_n	sign of $f(a)f(x_n)$
1	0	1	0.5	+
2	0.5	1	0.75	_
3	0.5	0.75	0.6258	-
4	0.5	0.625	0.5625	+
5	0.625	0.625	0.5937	

According the above table root of $f(x) = x^2 + x - 1$ in [0, 1] approximately is 0.5937.

Example 3

Suppose we are going to approximate the root of following equation by the bi-section algorithm until $|f(x_n)| < 0.01$.

$$f(x) = x^2 - (1-x)^5 = 0$$

In fact we want finding root of function $f(x) = x^2 - (1-x)^5$.

Now we guess two number such as a,b so that $f(a)f(b) < 0$. Let a=0 and b=1 then f(a)=-1 and f(b)=1 therefore

$$f(a)f(b) = (-1)\times 1 < 0.$$

Because f is polynomial then it is continuous in every interval of real numbers particularly in [0, 1]. Therefore f has conditions of theorem 1 then f has at least one root in [0, 1].

Derivative of f equal to:

$$f'(x) = 2x + 5(1-x)^4$$

Obviously $f'(x)$ is positive in (0, 1) therefore f has conditions of theorem 2 too. Namely f(x) =0 has at most a root in (0, 1).

According to the theorem 1 and theorem 2, f(x) =0 has just one root in (0, 1). Now we con use the bi-section algorithm to find the root of $f(x) = x^2 - (1-x)^5$ in interval [0, 1].

The following table is summary of the bi-section method to solve this example at the five iterations:

Table 2. Stages of bi-section algorithm-Example 3.

Iterations	a	b	x_n	sign of $f(a)f(x_n)$	$\|f(x_n)\|$
1	0	1	0.5	_	0.2167
2	0	0.5	0.25	+	0.1748
3	0.25	0.5	0.375	_	0.0452
4	0.25	0.375	0.3125	+	0.0559
5	0.3125	0.375	0.3437	+	0.0035

According the above table root of $f(x) = x^2 - (1-x)^5$ in [0, 1] approximately is 0.3437.

4. Computational Results

To illustrate bi-section algorithm, two standard examples will be solved in this section.

Example 4 [17]

Consider the following linear bi-level programming problem:

$$\min \mathrm{x} - 4\mathrm{y}$$

$$\mathrm{s.t} \min y$$

$$\mathrm{s.t} \;\; \mathrm{x} + \mathrm{y} \geq 3,$$

$$-2\,\mathrm{x} + \mathrm{y} \leq 0,$$

$$2\,\mathrm{x} + \mathrm{y} \leq 12,$$

$$3\mathrm{x} - 2\mathrm{y} \leq 4,$$

$$\mathrm{x}, \mathrm{y} \geq 0.$$

Using KKT conditions, the following problem is obtained:

$$\min \mathrm{x} - 4\mathrm{y}$$

$$\mathrm{s.t} -\mu_1 + \mu_2 + \mu_3 - 2\mu_4 = -1$$

$$\mu_1(-\mathrm{x} - \mathrm{y} + 3) = 0,$$

$$\mu_2(-2x+y)=0,$$

$$\mu_3(2x+y-12)=0,$$

$$\mu_4(3x-2y-4)=0,$$

$$-x-y+3\le 0,$$

$$-2x+y\le 0,$$

$$2x+y-12\le 0,$$

$$3x-2y-4\le 0,$$

$$x,y,\mu_1,\mu_2,\mu_3,\mu_4\ge 0.$$

This problem have been solved using bi-section algorithm, Optimal solution have been presented according to Table 3.

Table 3. Comparison of optimal solutions by hybrid algorithm-Example 4.

Best solution by our method		Best solution according to reference [17]		Optimal solution	
(x^*,y^*)	z^*	(x^*,y^*)	z^*	(x^*,y^*)	z^*
(4,4)	-12	(3.9,4)	-12.1	(4,4)	-12

Example 5 [17]

Consider the following linear bi-level programming problem.

$$\min 4x+y_1+y_2$$

$$\text{s.t} \min x+3y_1$$

$$\text{s.t}\ \ x+y_1+y_2\ge \frac{25}{9},$$

$$x+y_1\le 2,$$

$$y_1+y_2\le \frac{8}{9},$$

$$x,y_1,y_2\ge 0.$$

After applying KKT conditions and smoothing method, the above problem will be transformed into the problem which will be solved using bi-section algorithm. Optimal solution for this example is presented according to Table 4.

More problems with deferent sizes have been solved by two approaches and computation results have been proposed in Table 5. The programming is performed using MATLAB 7.1 and use a personal computer (CPU: Intel (R) Celeron(R) 1000 M @ 1.8 GHz, RAM: 4 GB) to execute the program. It is easy to see that TA algorithm is better than HA according to Table 6.

Table 4. Comparison of optimal solutions by hybrid algorithm-Example 5.

Best solution by our methodz*		Best solution according to reference [17]z*		Optimal solutionz*	
(x^*,y_1^*,y_2^*)	z^*	(x^*,y_1^*,y_2^*)	z^*	(x^*,y_1^*,y_2^*)	z^*
(1.88.,0.79,0.09)	8.40	(1.83,0.89,0.004)	8.21	$(\frac{17}{9},\frac{8}{9},0)$	8.44

Table 5. Comparison of optimal solutions and elapsed time with different examples 6-9 of BLPP by Taylor algorithm

	Best solution by our method with different values of ε	Best solution according to reference [3,7,26,27]
	O.S	O.S
Example 6	(1.33,1.30,0.03,0.37,1.30,0.91) $F^*=-52.41$	(1.32,1.28,0,0.33,1.25,0.92) $F^*=-51.31$
Example 7	(2,0,0,0) $F^*=10$	(2,0,0,0) $F^*=10$
Example 8	(0,1,0.2,0.6,0.4) $F^*=30.3$	(0,0.9,0,0.6,0.4) $F^*=29.2$
Example 9	(17,11.04) $F^*=86.08$	(17.45,10.90) $F^*=85.08$

5. Conclusion and Future Work

The main difficulty of the multi-level programming problem is that after using the KKT conditions the non-linear constraints are appeared. In this paper was attempted to remove these constraints by the proposed theorem, slack variables and proposed PSOMGA algorithm. As mentioned previously the authors have been combined two continuous and discrete effective approaches to the linear BLPP which this form of combining has not been studied by any researchers. According to the Tables the proposed method presents optimal solution in appropriate time and iterations. In the future works, the following should be researched:

(1) Examples in the larger sizes can be supplied to illustrate the efficiency of the proposed algorithm.

(2) Showing the efficiency of the proposed algorithm for solving other kinds of BLPP such as quadratic and non-linear.

(3) Solving other kinds of multi-level programming problem such as tri-level programming problem.

Nomenclature

$F_1(x,y,z)$	Objective function of the first level in the TLPP
$F_2(x,y,z)$	Objective function of the second level in the TLPP
$F_3(x,y,z)$	Objective function of the third level in the TLPP
$g(x,y,z)$	Constraints in the TLPP
w	Slack variable

v	Slack variable
$F(x,y)$	Objective function of the first level in the BLPP
$f(x,y)$	Objective function of the first level in the BLPP
$g(x,y)$	Constraints in the BLPP
S	A nonempty convex set
L	Lagrange function
α	Lagrange Coefficient
β	Lagrange Coefficient
μ	Lagrange Coefficient
P	Initial population
P'	Crossover population
P''	Mutation population
Q	Set of chromosomes in the current generation
(x^*, y^*, z^*)	Optimal solution for the TLPP
(x^*, y^*)	Optimal solution for the BLPP

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